Lab/UFR-HEP0512/GNPHE/0514/VACBT/0514

# Topological SL (2) Gauge Theory on Conifold

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February 1, 2008

**Abstract**

Using a two component $SL(2)$ isospinor formalism, we study the explicit link between conifold $T^*\mathbb{S}^3$ and q-deformed non commutative holomorphic geometry in complex four dimensions. Then, thinking about conifold as a projective complex three dimension hypersurface embedded in non compact $WP^5(1,-1,1,-1,1,-1)$ space and using conifold local isometries, we study topological $SL(2)$ gauge theory on $T^*\mathbb{S}^3$ and its reductions to lower dimension sub-manifolds $T^*\mathbb{S}^2$, $T^*\mathbb{S}^1$ and their real slices. Projective symmetry is also used to build a supersymmetric $\mathrm{QFT}_4$ realization of these backgrounds. Extensions for higher dimensions with conifold like properties are explored.

**Key words**: Conifold, q-deformation, non commutative complex geometry, topological gauge theory. Nambu like background.

# Contents



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# 1 Introduction

In the last decade there has been an intensive interest to supersymmetric field theories embedded in $10D$ type II superstring models on Calabi-Yau manifolds. Studies involving conifold backgrounds have been shown particularly interesting and are of basic importance. They are behind the derivation of many results in superstring compactification and brane physics. It is worthwhile to recall the correspondence between conifold and two dimensional $c = 1$ non critical string with cosmological term [1]-[5], conifold transitions, branes and fluxes, open/closed string duality [6]-[10]; and recent development in topological string theory and non commutative geometry [11]-[17].

Motivated by similarities with non commutative Chern-Simons gauge theory on 3-sphere and fractional quantum Hall fluids in higher dimensions, we consider in this paper conifold local isometries and use $SL(2)$ isospinor formalism to study non commutative topological gauge theory on $T^*\mathbb{S}^3$ and reductions to its sub-manifolds $T^*\mathbb{S}^2$ and $T^*\mathbb{S}^1$ as well as their compact real slices. To that purpose, we first explore the relation between conifold and non commutative geometry. This link, which is already visible at the level of conifold defining equation $x_1y_2 - x_2y_1 = \mu$, may be used to develop new perspectives in non commutative gauge theory. A typical example in this matter is given by the D string fluid model studied in [39] and which generalizes FQH systems in Laughlin state with filling fraction $\frac{1}{k}$. Then, thinking about conifold as a projective complex three dimension hypersurface embedded in non compact $WP^5(1, -1, 1, -1, 1, -1)$, we develop an explicit method to derive non commutative holomorphic topological gauge theory on conifold with local $SL(2)$ isometry as gauge group. This topological field theory has the remarkable property of extending Chern-Simons gauge theory on the 3-sphere. But before describing the organisation of this study and go into technical details, it is interesting to give other motivations behind this study. These are given by the three following:

(**1**) For the link between conifold and non commutative geometry, one may think about conifold defining equation,

$$x_1y_2 - x_2y_1 = \mu, \tag{1.1}$$

with complex modulus $\mu$, as a typical q-deformed relation of non commutative geometry (NC) [18]-[23]. This equation supplemented by the obvious ones $[x_i, x_j] = 0$ and $[y_i, y_j] = 0$, which read altogether in $SL(2)$ covariant form as $\varepsilon^{ij}x_iy_j = \mu$, $\varepsilon^{ij}x_ix_j = 0$, $\varepsilon^{ij}y_iy_j = 0$, can be equally put in the form,

$$x_{[i}y_{j]} = \vartheta_{ij}, \qquad x_{[i}x_{j]} = 0, \qquad y_{[i}y_{j]} = 0, \qquad i, j = 1, 2, \tag{1.2}$$

with a constant " complex magnetic field" $\vartheta_{ij} \sim \mu\varepsilon_{ij}$ and where $[ij]$ refers to usual

antisymmetrisation of indices. Setting $x_i = Z_{1i}$ and $y_j = Z_{2j}$, the above relations combine as $Z_{ki}Z_{lj} - Z_{kj}Z_{li} = \varepsilon_{kl}\vartheta_{ij}$ and may be read also as,

$$Z_{ki}Z_{lj} - \mathcal{R}_{kl}^{mn} Z_{mj}Z_{ni} = \varepsilon_{kl}\vartheta_{ij}, \tag{1.3}$$

with $\mathcal{R}_{kl}^{mn} = \varepsilon_k^m \varepsilon_l^n$. Equation $x_1y_2 - x_2y_1 = \mu$ is then just the unique non trivial relation of complex $2 \times 2$ matrix coordinate system. With such a formulation, one disposes of an other picture of thinking about conifold; and so one can borrow techniques and results on q-deformed non commutative geometry and symplectic manifolds to build new representations for conifold and its sub-manifolds. This view offers as well a new way to look for geometric extensions with conifold like features type the symplectic varieties with $SP(n)$ isometries and Nambu like geometry considered in discussion section.

(**2**) Conifold geometry seen as a projective hypersurface may be used to construct supersymmetric QFT$_4$ realizations embedded in type II superstring on conifold. Recall that from super QFT$_4$ view, the projective gauge invariance is the abelian gauge sub-symmetry in supersymmetric quiver gauge theories. However by describing conifold using the equation $x_1y_2 - x_2y_1 = \mu$, one is in fact thinking about it as a complex three dimension holomorphic hypersurface embedded in complex four dimension space $\mathbb{C}^4$ where projective symmetry is fixed. To implement projective invariance, one needs to go beyond $\mathbb{C}^4$; for instance to four complex dimension non compact projective spaces $W\mathbb{P}^4(-1,1,-1,1,-1)$ where $\mathbb{C}^4$ appears as a local patch described by the projective gauge fixing $\sigma = 1$. The extra variable $\sigma$ captures the projective symmetry of $W\mathbb{P}^4$ space; that is $\sigma \equiv \lambda\sigma$ with the usual projective parameter $\lambda \in \mathbb{C}^*$. Relaxing the condition $\sigma = 1$ to arbitrary values $\sigma \in \mathbb{C}^*$ and imposing projective invariance, one can easily get the projective hypersurface describing conifold geometry; but this time embedded in $W\mathbb{P}^4(q_\sigma, q_x, q_y, q_z, q_w)$. A quick way to determine the projective weights $(q_\sigma, q_x, q_y, q_z, q_w)$ is to start from eq(**??**) and rewrite it as

$$\left(\frac{x_1}{\sigma}\right)(\sigma y_2) - (\sigma x_2)\left(\frac{y_1}{\sigma}\right) = \mu. \tag{1.4}$$

Renaming the variables as $x = \frac{x_1}{\sigma}$ and so on, we end with the hypersurface $xy - zw = \mu$ describing the usual $T^*\mathbb{P}^1$ fibration over $\mathbb{C}^*$ embedded in the non compact projective space $W\mathbb{P}^4(-1,1,-1,1,-1)$. To keep Calabi-Yau condition manifest, one should go a step further beyond $W\mathbb{P}^4(-1,1,-1,1,-1)$ and think about above relation as given by the complex three dimension hypersurface,

$$\begin{aligned}(\sigma_+ x_1)(\sigma_- y_2) - (\sigma_- x_2)(\sigma_+ y_1) &= \mu, \\ \sigma_+\sigma_- &= 1,\end{aligned} \tag{1.5}$$

embedded in $W\mathbb{P}^5(1,-1,1,-1,1,-1)$. This formulation is very instructive as it allows to make an idea on the super QFT$_4$ realization of this background where $x$, $y$, $z$ and $w$

are respectively associated with the moduli of fundamental matter superfields $X_+$, $Y_-$, $Z_+$ and $W_-$. The $\pm$ sub-indices refer to the projective charge carried by the superfields. Concerning the extra variables $\sigma_\pm \in \mathbb{C}^*$, they are associated with two chiral superfields $\Sigma_-$ and $\Sigma_+$ constrained as $\Sigma_-\Sigma_+ = 1$. In this picture, the usual neutral adjoint matter chiral superfield $\Phi$ of supersymmetric quiver gauge theories appears as the Lagrange superfield implementing the constraint eq $\Sigma_-\Sigma_+ = 1$ in the holomorphic field action.

(**3**) The other motivation deals with the relation between conifold geometry and non commutative topological $SL(2)$ gauge theory. Notice that under local change $x_i \rightarrow \Upsilon_i^k y_k$ and $y_j \rightarrow (\Upsilon^{-1})_j^l x_l$ with $\Upsilon = \Lambda(x_i, y_i)$ and $\det \Upsilon = 1$, conifold eq $\varepsilon^{ij} x_i y_j = \mu$ remains invariant. On the moduli space of the supersymmetric $\mathrm{QFT}_4$ vacua, these isometries correspond to a non commutative topological holomorphic $SL(2, \mathbb{C})$ gauge theory on conifold. As we will see in section 6, the gauge field constraint eqs are also the field equations of motion of the topological $SL(2, \mathbb{C})$ gauge theory. This result is obviously valid in the projective representation where conifold is thought of as a complex three hypersurface embedded in $W\mathbb{P}^5(1, -1, 1, -1, 1, -1)$. The novelty is that in present case, we have a special abelian sub-isometry which may be used to get more insight in the huge topological $SL(2, \mathbb{C})$ gauge theory and its reductions to the lower dimension holomorphic gauge theories on $T^*\mathbb{S}^2$ and $T^*\mathbb{S}^1$ as well as on real slices. In the abelian sector, the usual global projective symmetry, $\sigma \rightarrow \lambda\sigma$ and so on, get promoted to a gauge symmetry $\sigma \rightarrow \Lambda\sigma, \ldots$, with $\Lambda = \Lambda(\sigma, x, y)$. By focusing on this abelian gauge sub-symmetry, we show how non commutative topological $\mathbb{C}^*$ gauge theory follows from a simple gauge principle relying on equating the global $SL(2)$ algebra,

$$[D_+, D_-] = 2D_0, \qquad [2D_0, D_\pm] = \pm 2D_\pm, \tag{1.6}$$

with the corresponding gauge covariant one namely,

$$[\mathcal{D}_+, \mathcal{D}_-] = 2\mathcal{D}_0, \qquad [2\mathcal{D}_0, \mathcal{D}_\pm] = \pm 2\mathcal{D}_\pm. \tag{1.7}$$

Here the $\mathcal{D}_{0,\pm}$'s are the covariant derivatives and read as $\mathcal{D}_{0,\pm} = D_{0,\pm} - A_{0,\pm}$. Non commutative topological holomorphic gauge theory on conifold follows naturally as a solution of these constraint eqs. Chern Simons gauge theory on $\mathbb{S}^3$ follows as well by imposing reality condition.

Along with these motivations, it is interesting to note moreover that above mentioned conifold features have similar ones in quantum Hall systems and attractor mechanism of Hartlee-Hawking wave function for flux compactifications [24, 25]. The non commutative topological gauge theory for conifold may be related to Susskind proposal on quantum Hall (QH) systems [26]. Restricting conifold $T^*\mathbb{S}^3$ to its real three dimensional slice, the $U(1)$ Chern-Simons (CS) gauge theory on $\mathbb{S}^3$ may be, roughly speaking,

compared with the $(2+1)$ non commutative CS gauge theory describing fractional QH systems. This formal similarity is even more striking since there is also a correspondence between Susskind model for Laughlin state with filling fraction $\nu = \frac{1}{k}$, with $k$ positive (odd) integer, and the attractor mechanism of Hartlee-Hawking universe wave function on $\mathbb{S}^3$ fixing the global complex deformation parameter of the conifold as $\mu = k + i\frac{\phi}{\pi}$ with $\phi$ a real modulus [27].

The organisation of this paper is as follows: In section 2, we review aspects of conifold as a hypersurface embedded in the ambiant complex space $\mathbb{C}^4$, a matter to fix the ideas and convention notations. In section 3, we show that conifold embedded in $\mathbb{C}^4$ may be also viewed as a q-deformed non commutative complex four dimension holomorphic geometry with a very special antisymmetric field $\vartheta_{il}$. The same result is valid for $\mathbb{S}^3$ embedded in $\mathbb{R}^4$ and the other sub-manifolds. In section 4, we develop a conifold representation using a complex three dimension projective hypersurface embedded in $WP^5(1,-1,1,-1,1,-1)$ and give a super $\mathrm{QFT}_4$ realization. In section 5, we study conifold diffeomorphism using two kinds of coordinate frames and in section 6, we consider the derivation of non commutative topological gauge theory by focusing on the $\mathbb{C}^*$ model. In section 7, we give a conclusion and make discussions regarding higher dimension extensions.

## 2 Conifold as a $\mathbb{C}^4$ hypersurface

From the point of view of algebraic geometry, complex three dimension conifold $T^*\mathbb{S}^3$ with a global complex deformation parameter $\mu$ is generally defined as a hypersurface $H_0 = H_0(x_1, x_2, y_1, y_2)$ embedded in the four complex space $\mathbb{C}^4$ as,

$$H_0 : x_1 y_2 - x_2 y_1 = \mu, \tag{2.1}$$

where $x_1, x_2, y_1, y_2$ are complex holomorphic coordinates with the unique restriction given above. Such a relation, which is singular for $\mu = 0$ and corresponding to a shrinking real three sphere, appears as the topological ground ring of two dimensional non critical $c = 1$ string theory [1] and has a set of isometries from which one can extract precious informations. To exhibit explicitly these isometries, it is interesting to go to the $2 \times 2$ matrix coordinates representation by using the correspondence $\mathbb{C}^4 \sim Mat(2, \mathbb{C})$ and redefine conifold hypersurface eq $H_0$ as given by the determinant of a complex holomorphic $2 \times 2$ matrix $Z$, i.e

$$\det Z = (x_1 y_2 - x_2 y_1) = \mu, \tag{2.2}$$

with,

$$Z=\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}, \qquad Z.Z^{-1}=I. \tag{2.3}$$

This complex holomorphic matrix representation, which breaks down for $\mu=0$, has a manifest $GL(2,\mathbb{C})\sim\mathbb{C}^{*}\times SL(2,\mathbb{C})$ automorphism symmetry acting through changes generated by the following arbitrary $M$ matrix,

$$Z\rightarrow MZM^{-1}. \tag{2.4}$$

Strictly speaking, there are two main options for thinking about this matrix $Z$; either as a matrix operator acting on an underlying complex two dimension space $\mathbb{C}^2$, or as pure matrix coordinates $Z_{ij}$ parameterizing $\mathbb{C}^4$. If forgetting about small details, the two options are a priori equivalent and the apparent differences is linked with the way they are handled. Keeping this in mind, let us focus for the moment on the isometries of the hypersurface $H_0$. Since the $\mathbb{C}^*$ factor is an abelian symmetry, conifold isometries seems at first sight given by global $SL(2,\mathbb{C})$ symmetries. However, this $SL(2,\mathbb{C})$ isometry is just the global part of a huge gauge symmetry generated by conifold diffeomorphisms $Diff\left(T^{*}S^{3}\right)$ typically captured by local matrices as shown below,

$$M_{ij}=M_{ij}\left(x_1,x_2,y_1,y_2\right), \qquad i,j=1,2. \tag{2.5}$$

As there is no differential operators $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$ in the conifold defining eq $x_1y_2-x_2y_1=\mu$, it does not matter whether $M$ is a constant matrix or depending on the local variables $x_i$ and $y_i$. Leaving these details for later, note that invariance of the conifold hypersurface $H_0$ follows from the property of the det mapping which acts on $Z$ as

$$\det MZM^{-1}=\det Z, \tag{2.6}$$

where $M$ stands for an arbitrary $GL(2,C)$ gauge transformation matrix. Note that for gauge transformations restricted to $K$ sitting in $SL(2,C)$, any change of $Z$ type

$$Z_{ij}\rightarrow Z'_{ij}=K_{il}Z_{lj}, \qquad K\in SL(2,C) \tag{2.7}$$

is also a symmetry of the conifold; thanks to the property $\det(MZ)=(\det K)(\det Z)$. Note also that with the change $Z'=KZ$ , the previous trivial abelian factor $\mathbb{C}^*\simeq GL(2)/SL(2)$ eq(2.6) is no longer a conifold symmetry of eq(2.7); it appears as a "scaling transformation" which has much to do with the scaling symmetry used in [28] to study local complex deformations of conifold dealing with the building of $\mathbb{S}^3$ quantum cosmology. Recall that these local complex are known to model momenta and winding corrections in $c=1$ non critical string; for details see [29]. Since the main difference between the transformations $Z'=MZM^{-1}$ and $Z''=KZ$ is this $\mathbb{C}^*$ abelian scaling

factor, we shall drop it in what follows. Nevertheless it is interesting to note here that there is a second $\mathbb{C}^*$ symmetry that we will encounter below and which turns out to play an important role. It comes from the factorisation of $SL(2)$ as the product of $\mathbb{C}^*$ with the complex holomorphic coset $SL(2)/\mathbb{C}^*$; then it should not be confused with $\mathbb{C}^* \simeq GL(2)/SL(2)$ we have just disregarded.

In the $Z_{ij}$ matrix coordinates formalism, conifold symmetries are then manifestly exhibited. But this is not all the story; along with this useful property, one discovers moreover a set of basic features that pass under heard in the usual $(x_1, x_2, y_1, y_2)$ component formalism. These features captures essential data on conifold geometry and have interesting physical interpretations. In what follows, we study three of the special conifold features that seem to us of basic importance in the understanding of the structure of the field theoretical models relying on conifold geometry. These features concern the following points:

(i) Working explicitly the link between conifold, together with its sub-manifolds $T^*\mathbb{P}^1$, $\mathbb{S}^3$ and $\mathbb{S}^2$, and q deformed non commutative complex four dimension holomorphic geometry. As it will be explicated later, these geometries are very special in the sense that their quadratic algebraic geometry equations look like the q deformed canonical commutation relations of quantum physics opening then issues for wider applications. Focusing on conifold equation, it is not difficult to check that $\det Z = \varepsilon^{ik}\varepsilon^{jl} Z_{ij} Z_{kl} = \mu$ is equivalent to the specific q-deformed non commutative geometry relations,

$$Z_{ki} Z_{lj} - Z_{kj} Z_{li} = \varepsilon_{kl} \vartheta_{ij}, \tag{2.8}$$

where $\mu$ is as before and where $\varepsilon_{ik}$ is the usual invariant two dimensional antisymmetric tensor.

(ii) Use the link $T^*\mathbb{S}^3 \sim \mathbb{C}^* \times T^*\mathbb{P}^1$ between conifold $T^*\mathbb{S}^3$ and cotangent bundle on the projective space $\mathbb{P}^1$ to re-formulate conifold geometry as a complex three dimension projective hypersurface in non compact space $W\mathbb{P}^5(1,-1,1,-1,1,-1)$ with complex coordinates $(\sigma_+, \sigma_-, x, y, z, w)$ and a $\mathbb{C}^*$ projective symmetry as,

$$(\sigma_+, \sigma_-, x, y, z, w) \to \left(\lambda \sigma_+, \frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda} y, \lambda z, \frac{1}{\lambda} w\right). \tag{2.9}$$

This representation plays a fundamental role in constructing supersymmetric quiver gauge theories with a $U(1)$ sub-symmetry. There, the complex $(\sigma_+, \sigma_-, x, y, z, w)$ variables describe superfields moduli minimizing the chiral superpotential,

$$\begin{aligned} \int d^2\theta \mathcal{W} &= \int d^2\theta \left([X_+ \Phi_0 Y_- - Z_+ \Phi_0 W_-] - \mu \Sigma_+ \Phi_0 \Sigma_-\right) \\ &\quad + \int d^2\theta \Psi_0 \left(\Sigma_+ \Sigma_- - 1\right) \\ &\quad + \int d^2\theta \mathrm{W}\left(\Sigma_\pm, X_+, Y_-, Z_+, W_-\right), \end{aligned} \tag{2.10}$$

with respect to the neutral chiral matter superfields $\Phi_0$ and $\Psi_0$. The sub-indices $\pm$ refer to the gauge charges. The naive correspondence would be associating the $\sigma_+$, $\sigma_-$, $x$, $y$, $z$ and $w$ variables with the VEVs of corresponding superfields $(\Sigma_-, X_+, Y_-, Z_+, W_-)$. A priori we may have $x =< X_+ >$, $y =< Y_- >$, $z =< Z_+ >$, $w =< W_+ >$ and $\sigma_\pm =< \Sigma_\pm >$ following from extremizing $\mathrm{W}\left(\Sigma_\pm, X_+, Y_-, Z_+, W_-\right)$,

$$d\mathrm{W}\left(\Sigma_\pm, X_+, Y_-, Z_+, W_-\right) = 0. \tag{2.11}$$

However gauge invariance requires that the exact picture should be as follows

$$\begin{aligned} xy &= < X_+ Y_- >, \qquad zw =< Z_+ W_- >, \\ \sigma_- x &= < \Sigma_- X_+ >, \qquad \sigma_- w =< \Sigma_- Z_+ >, \\ \sigma_+ y &= < \Sigma_+ Y_- >, \qquad \sigma_+ w =< \Sigma_+ W_- > \end{aligned} \tag{2.12}$$

More details on this method are given in subsection 4.2. For a quite similar super $\mathrm{QFT}_4$ analysis dealing with $T^*\mathbb{P}^1$ background; see [30, 9].

(iii) Referring to above superfield theoretical interpretation and to conifold eq $x_1 y_2 - x_2 y_1 = \mu$, which we can usually rewrite it as follows,

$$x_1 y_2 - x_2 y_1 = \left(\frac{x_1}{\sigma}\right)(\sigma y_2) - \left(\frac{x_2}{\sigma}\right)(\sigma y_1) = \mu, \tag{2.13}$$

for any non zero complex modulus $\sigma$. This change corresponds to moving from the $\mathbb{C}^*$ invariant coordinate frame $(x_1, x_2, y_2, y_1)$ to the projective one $(\sigma, x, y, z, w)$. The objective of this part of the analysis is to extend the global change (2.9) to local gauge transformations

$$\sigma \to \frac{1}{\Lambda}\sigma, \qquad \Lambda = \Lambda\left(\sigma, x, y, z, w\right), \tag{2.14}$$

which are still isometries of conifold. This local change induces in turns,

$$\begin{aligned} x &= \left(\frac{x_1}{\sigma}\right) \to \Lambda\left(\frac{x_1}{\sigma}\right) = \Lambda x \\ z &= \left(\frac{x_2}{\sigma}\right) \to \Lambda\left(\frac{x_2}{\sigma}\right) = \Lambda Z \\ y &= (\sigma y_1) \to \frac{1}{\Lambda}(\sigma y_1) = \frac{1}{\Lambda} y \\ w &= (\sigma y_2) \to \frac{1}{\Lambda}(\sigma y_2) = \frac{1}{\Lambda} w. \end{aligned} \tag{2.15}$$

Then study the gauge theory behind this gauge invariance principle. As we will prove in section 5, this is a non commutative holomorphic topological gauge theory which on real slice, reduces to the non commutative topological Chern-Simons gauge theory on the three sphere. This topological gauge theory deals with abelian isometries and is in fact a part of the huge $SL\left(2\right)$ holomorphic gauge theory.

# 3 Conifold as a q-deformed NC $\mathbb{C}^4$ geometry

In this section, we first introduce the two component formalism to parameterize conifold geometry. Then, we discuss its link with q-deformed non commutative geometry in $\mathbb{C}^4$. Finally we give a classical mechanical like model realizing conifold background. This complex holomorphic model is inspired from similarities with the classical dynamics of quantum Hall particles moving in a strong magnetic field.

## 3.1 Two component formalism

As far as conifold defining eq(2.1) is concerned, one learns from the coordinate matrix representation that if we insist on using component formalism for $T^*\mathbb{S}^3$, the natural way to do it is by using a two component formalism involving the two complex holomorphic $SL(2,C)$ isospinors,

$$u_i = (x_1, x_2), \qquad v_i = (y_1, y_2). \tag{3.1}$$

These two component variables are given by the rows of the matrix coordinate $Z_{ij}$; that is $u_i = Z_{1i}$ and $v_i = Z_{2i}$. In terms of these isospinor variables, conifold constraint eq reads as

$$\begin{aligned} \varepsilon^{ij} u_i v_j &= \mu, \\ \varepsilon^{ij} u_i u_j &= 0, \\ \varepsilon^{ij} v_i v_i &= 0, \end{aligned} \tag{3.2}$$

where $\varepsilon^{ij}$ is the usual two dimensional antisymmetric invariant tensor with $\varepsilon^{12} = 1$ and inverse $\frac{1}{2}\varepsilon_{ji}$. The first relation $\varepsilon^{ij} u_i v_j = \mu$ expresses just $SL(2,C)$ invariance of conifold hypersurface in $\mathbb{C}^4$. The two remaining others rests on the property that $u_i$ and $v_i$ are commuting bosonic isodoublets in same manner as for Penrose twistors [31]. The fact that conifold holomorphic hypersurface $H_0$ takes the above form seems at first sight something obvious and it is just a way of exhibiting manifestly $SL(2,C)$ global isometries. This is true, but there is something more. The idea is that, by help of the inverse of $\varepsilon^{ij}$; i.e $\varepsilon^{ij}\varepsilon_{ji} = 2$, these relations may be also put into the following remarkable form,

$$\begin{aligned} u_i u_j - u_j u_i &= 0, \\ u_i v_j - v_i u_j &= \vartheta_{ij}, \\ v_i v_j - v_j v_i &= 0, \end{aligned} \tag{3.3}$$

where we have set $\vartheta_{ij} = \frac{\mu}{2}\varepsilon_{ji}$. But these are familiar relations in non commutative geometry; the only differences are that in present case we are dealing with complex

holomorphic analysis in higher dimensions and that the precise interpretation is that the identity $(u_i v_j - v_i u_j) = \vartheta_{ij}$ is q deformed relation

$$u_i v_j - R_{ij}^{kl} v_k u_l = \vartheta_{ij}, \tag{3.4}$$

with $R_{ij}^{kl} = \varepsilon_i^k \varepsilon_j^l$. Forgetting about this technical detail, the two component isospinor formalism we have been introducing establishes therefore a direct and manifest link between conifold hypersurface and q-deformed non commutative holomorphic geometry in complex four dimensions with magnitude of the deformation tensor given by the global complex deformation $\mu$.

**Theorem 1** *Conifold $T^*\mathbb{S}^3$ with complex moduli $\mu$ is equivalent to a q-deformed non commutative complex four dimension geometry with $SL(2)$ isometry and holomorphic magnetic field $B_{IK} = -\frac{\mu}{2}\varepsilon_{ik}\varepsilon_{jl}$, $I = (i,j)$ and $K = (k,l)$. In this view, the singular limit $\mu = 0$ corresponds to commuting $\mathbb{C}^4$. This result is also valid for the complex two dimension holomorphic sub-manifold $T^*\mathbb{P}^1$ having a $SL(2)/\mathbb{C}^*$ isometry, the real slice $\mathbb{S}^3$ with $SU(2)$ isometry and the two sphere $\mathbb{S}^2$ with symmetry $SU(2)/U(1)$.*

With this result at hand, one may be tried to do something with; starting with the search for bonds with relevant quantities in type II superstring compactifications on Calabi-Yau manifolds and brane physics. A way to make contact with the real quantities is to restrict conifold $T^*\mathbb{S}^3$ to its real slice obtained by setting $y_2 = \overline{x_1}$ and $y_1 = \overline{x_2}$. The defining equation of the resulting real three sphere $\mathbb{S}^3$ embedded in $\mathbb{C}^2 \sim \mathbb{R}^4$ reads then as,

$$|x_1|^2 + |x_2|^2 = \operatorname{Re}\mu = p, \tag{3.5}$$

where now the real number $p$ is the radius squared of $\mathbb{S}^3$. In this real restriction, the isospinor $v_i$ get identified with $\overline{u}_i = \left(\overline{u^i}\right)$, the complex conjugate of $u^i = \varepsilon^{ij} u_j$ and eqs(3.3) reduce to the following special non commutative geometry relations,

$$\begin{aligned} u_i u_j - u_j u_i &= 0, \\ u_i \overline{u}_j - \overline{u}_i u_j &= p\varepsilon_{ij}, \\ \overline{u}_i \overline{u}_j - \overline{u}_j \overline{u}_i &= 0. \end{aligned} \tag{3.6}$$

If we forget about reality and continue to work with complex holomorphic quantities, one may be tempted to derive new representations for conifold geometry by mimicking standard analysis in quantum mechanics and non commutative geometry developed in literature [32]-[33]. The global complex deformation parameter $\mu$ has formally a similar role as Planck constant $\hbar$ of quantum mechanics and so may be used for quasi-classical studies using formal series in powers of $\mu$ as,

$$\mathcal{F}(Z_{ij}) = \sum_n \mu^n F_n(Z_{ij}), \tag{3.7}$$

where the $F_n(Z_{ij})$'s refer to the *n-th* perturbation order of correction terms. This expansion is in agreement with the structure of the expansion of free energy $\mathcal{F}_{top}$ of the B model topological string on locally conifold with local complex deformations.

## 3.2 A classical mechanical like model

Letting the two component variables $u$ and $v$ to have a time dependence as $u^i = U^i(t)$ and $v_i = V_i(t)$ with $\partial_t U^i$ being the "time" derivative, this pair of isospinors may be also interpreted as canonical variables in a complexified dynamical Lagrangian description,

$$V_i \sim \frac{\partial \mathcal{L}}{\partial(\partial_t U^i)}, \tag{3.8}$$

where $\mathcal{L} = \mathcal{L}(U, \partial_t U)$ is some holomorphic Lagrangian field density. The simplest example is given by the holomorphic field density,

$$\mathcal{L} \sim \frac{1}{\mu}\varepsilon^{ij} V_i \partial_t U_j = \frac{1}{\mu} V_i \partial_t U^i, \tag{3.9}$$

which may be thought of in the same lines as the real lagrangian describing a two dimensional quantum Hall effect particle moving in an external perpendicular and strong magnetic field $B \sim \frac{1}{\mu}$. Computing the conjugate momentum $P_i = \frac{\partial \mathcal{L}}{\partial(\partial_t U^i)}$ of the field variable $U^i$ by using eqs(3.8-3.9), we find

$$P_i = \frac{1}{\mu} V_i, \tag{3.10}$$

which up to using the q deformed canonical commutation relation $P_i U^i - U_i P^i \sim 1$, we get

$$\left(V_i U^i - U_i V^i\right) \sim \mu \tag{3.11}$$

which is nothing but conifold equation. One may also consider holomorphic hamiltonian like representations with,

$$\mathcal{H}(U, V) = \sum_{i,j=1}^{2} \varepsilon^{ij} V_i \partial_t U_j - \mathcal{L}, \tag{3.12}$$

to develop a symplectic like geometry. We believe that this two component isospinor formalism, which recalls Pauli two component spinor formalism of QED, may encode deeper informations on conifold. Together with the explicit non commutative property, this isospinor formalism opens a new insight for developing other conifold representations which may be linked with recent developments in topological string theory on conifold. Two of these representations, under investigation in [35], are given by the field theoretic description à la Susskind [27] and the the matrix model realization à la Susskind-Polychronakos [36, 37]. In the effective field theory approach, $\mathrm{Re}\,\mu$ is interpreted as

given by the inverse of an external constant magnetic field $B_{ex}$,

$$\mathrm{Re}\left(\vartheta_{ij}\right)=\frac{k}{B_{ex}}\varepsilon_{ij}, \qquad k=0,1,2,..., \tag{3.13}$$

and the used method follows Susskind philosophy in dealing with Quantum Hall Effect (QHE) as a non commutative effective Chern Simons $U(1)$ gauge theory. Recall in passing that in Susskind proposal [28] that non-Commutative Chern-Simons gauge theory on the (2+1) real space provides a natural framework to study the Laughlin state of filling factor $\nu=\frac{1}{k}$ with $k$ a positive (odd) integer. Following [28, ?], the non commutativity parameter $\vartheta$ of the co-moving plane coordinates is related to the filling factor $\nu$ and to the Chern-Simon effective field coupling constant $\lambda_{CS}$ as

$$\nu\times\vartheta\times B_{ex}=\nu\times\lambda_{CS}=1. \tag{3.14}$$

Upon on rescaling eq(3.13) as $\mathrm{Re}\left(\vartheta_{ij}\right)=k\varepsilon_{ij}$ and completing $\mathrm{Re}\left(\vartheta_{ij}\right)$ by switching on the imaginary part $\mathrm{Im}\,\vartheta=\frac{\phi}{\pi}$, we get a remarkable relation

$$\mu=k+\frac{i}{\pi}\phi, \qquad k=0,1,2,..., \tag{3.15}$$

which should be compared with one of two crucial relations derived in [?] and concerning attractor mechanism eqs for Hartlee Hawking universe wave function,

$$X^{i}=k^{i}+\frac{i}{\pi}\Phi^{i}, \tag{3.16}$$

where $k^{i}=\mathrm{Re}\left(X^{i}\right)$ are integers. In this equation, the complex numbers

$$X^{i}=\int_{A_i}\Omega_3 \tag{3.17}$$

are the usual complex structures given by integral of holomorphic 3-form $\Omega_3$ over 3-cycles $A_i$ of integral cohomology $H_3\left(T^{*}\mathbb{S}^3,\mathbb{Z}\right)$. The way Susskind model for quantum Hall systems is related to the attractor mechanism of [25] is still unclear for us; it needs more exploration.

To conclude this section, we would like to recall some facts. First note that appearance of non commutative geometry behavior for conifold is not a strange feature. It is quite well established that non commutativity lifts singularity [21]; and that deformed conifold has a non commutative geometry interpretation is then obvious. This property has been explicitly checked for Calabi-Yau orbifold geometries with discrete torsion and has been also interpreted in terms of fractional branes [23]. Note also that, even though not extensively explicited, non commutative behaviour of conifold is understood in the study of topological string theory on conifold; in particular by using geometric transition and mirror symmetry between $A$ and $B$ models. There, the complex parameter $\mu$ of $B$ model, given by integral of holomorphic 3-form $\Omega_3$, $\mu=\int_{A_3}\Omega_3$, over a 3-cycle $A_3$ of $H_3\left(T^{*}S^3\right)$, is mirror to complexified Kahler parameter $t=\int_{D_2}\left(B_2+iK_2\right)$ of the topological $A$ model involving magnetic like field.

# 4 Conifold as a projective hypersurface

In eq(2.1), conifold is viewed as a hypersurface embedded in $\mathbb{C}^4$ and the variables $x_1$, $x_2$, $y_1$ and $y_2$ were arbitrary complex holomorphic coordinates in $\mathbb{C}$ with the unique restriction $x_1y_2 - x_2y_1 = \mu$. From the point of view of super $\mathrm{QFT}_4$ functional analysis, the VEV's of the operator product $\mathcal{F}_1...\mathcal{F}_n$,

$$< \mathcal{F}_1...\mathcal{F}_n >= \int_{T^*\mathbb{S}^3} \prod_{i=1}^{2} DX_iDY_iD\Phi_0 \left(\mathcal{F}_1...\mathcal{F}_n\right) \exp \mathcal{S}\left[\Phi_0, X_i, Y_j\right], \tag{4.1}$$

with $\mathcal{F}\left(\Phi_0, X_i, Y_j\right)$ a generic function depending on the chiral superfields $\Phi_0, X_i$ and $Y_j$, the eq $x_1y_2 - x_2y_1 = \mu$ is recovered from the following global holomorphic superfield action,

$$\mathcal{S}\left[\Phi_0, X_i, Y_j\right] = \int d^4xd^2\theta \left(X_1\Phi_0Y_2 - X_2\Phi_0Y_1 - \mu\Phi_0\right) + \int d^4xd^2\theta W\left(X_i, Y_i\right) \tag{4.2}$$

Notice that while the superfields $X_i$ and $Y_j$ come in pairs, that is in $SL\left(2\right)$ doublets, the chiral superfield $\Phi_0$ is a singlet and appears as a Lagrange superfield parameter. This feature shows that $\mathcal{N} = 2$ supersymmetry spectrum (two hypermultiplets) is partially broken down to $\mathcal{N} = 1$ in agreement with known results on conifold geometries. Notice also that up to now, we have no gauge symmetry yet; the superfields $\Phi_0, X_i$ and $Y_j$ are not charged. In what follows, we study the gauging of this model by approaching conifold in an other way using projective symmetry to give gauge charges for superfields.

## 4.1 Implementing global projective symmetry

By help of conifold isometries, one may use the gauge transformation eq(2.4) to go to the matrix coordinate frame where the $2 \times 2$ matrix coordinate $Z$ is split into the product of two matrices $Y$ and $X$ as shown below,

$$Z = YX, \tag{4.3}$$

with

$$X = \begin{pmatrix} x & z \\ y & w \end{pmatrix}, \qquad Y = \begin{pmatrix} \sigma & 0 \\ 0 & \frac{1}{\sigma} \end{pmatrix}, \qquad \sigma \in \mathbb{C}^* \tag{4.4}$$

where, instead of the four complex variables $x_1$, $x_2$, $y_1$ and $y_2$, we have now five projective complex holomorphic variables $\left(\sigma, x, y, z, w\right)$ related to the previous ones like $x = x\left(\sigma, x_1\right), y = x\left(\sigma, y_1\right), z = z\left(\sigma, z_1\right), w = w\left(\sigma, w_1\right)$ as shown below,

$$\begin{aligned} x &= \frac{x_1}{\sigma}, \qquad z = \frac{x_2}{\sigma}, \\ y &= \sigma y_1, \qquad x = \sigma y_2. \end{aligned} \tag{4.5}$$

In these relations $\sigma \in \mathbb{C}^*$ and is a free complex variable capturing data on the projective abelian sub-symmetry $\mathbb{C}^*$ of the $SL(2,\mathbb{C})$ global isometry of conifold and where one recognizes the

$$xy - zw = \mu, \tag{4.6}$$

as the defining equation of $T^*\mathbb{P}^1$ geometry embedded in non compact $W\mathbb{P}^3(1,-1,1,-1)$. From the scaling $\sigma \to \frac{1}{\lambda}\sigma$, with $\lambda \in \mathbb{C}^*$, we get the projective transformations,

$$(\sigma, x, y, z, w) \to (\sigma', x', y', z', w') = \left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right), \tag{4.7}$$

of the non compact space $\mathbb{WP}^4(-1,1,-1,1,-1)$. Note that old coordinates $x_1$, $x_2$, $y_1$ and $y_2$ of $\mathbb{C}^4$ are recovered from $\sigma, x, y, z$ and $w$ by fixing projective symmetry which allows to set $\sigma = 1$ as in eqs(4.5). Note also that in a rigorous analysis, the Y matrix should be thought of as

$$Y = \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_- \end{pmatrix}, \qquad \sigma_-\sigma_+ = 1, \tag{4.8}$$

and the non compact $W\mathbb{P}^4(-1,1,-1,1,-1)$ is enlarged to $W\mathbb{P}^5(1,-1,1,-1,1,-1)$. In the $2\times2$ matrix coordinate frame, conifold equation $\det Z = \mu$ gets mapped to $\det X = \mu$ where, surprisingly there is no apparent dependence on the $Y$ matrix variable since $\det Y = 1$. Note that like before, the $SL(2)$ gauge transformations leaving stable conifold eq reads as

$$YX \to Y'X' = M(YX)M^{-1}, \tag{4.9}$$

or equivalently as $(MYM^{-1})(MXM^{-1})$ by inserting the $2\times2$ matrix identity $I = M^{-1}M$. Besides the factorisation $Z = YX$, we should also specify the way to deal with isometry eq (4.9). In the standard way, one thinks about the transformed matrix variables $Y'$ and $X'$ as given by the change $Y' = (MYM^{-1})$ and $X' = (MXM^{-1})$. The other possibility we will use below is to think about the transformation $Y'X' = MYXM^{-1}$ as associated with the naive change,

$$Y' = \Lambda Y, \qquad X' = X\Lambda^{-1}, \tag{4.10}$$

with $\Lambda$ an $SL(2,\mathbb{C})$ group matrix. To fix the ideas on the meaning of this transformation, let us consider the global $\mathbb{C}^*$ abelian sub-group of the conifold $SL(2,\mathbb{C})$ isometry by making the choice,

$$\Lambda = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \lambda \in \mathbb{C}^*, \tag{4.11}$$

and exhibit the meaning of the transformations (4.10) on the complex holomorphic coordinates $(\sigma, x, y, z, w)$. General expressions of the transformation $\Lambda$ has to do with

conifold diffeomorphisms; they will be discussed later. Restricting $SL(2,\mathbb{C})$ isometry to its abelian global part $\mathbb{C}^*$, then putting eq(4.11) back into eq(4.10), we re-discover eq(4.7). Therefore the coordinate mapping $Z = YX$ can be interpreted as moving from the coordinates $(x_1, x_2, y_1, y_2)$ of $\mathbb{C}^4$ to the coordinate frame $(\sigma, x, y, z, w)$ parameterizing $W\mathbb{P}^4(-1,1,-1,1,-1)$. In this frame, the conifold is a described by the invariant projective hypersurface,

$$F(\sigma, x, y, z, w) = xy - zw = \mu, \qquad \text{and} \qquad \sigma \text{ a free } \mathbb{C}^* \text{ variable}, \tag{4.12}$$

defining a $T^*\mathbb{P}^1$ fibration over $\mathbb{C}^*$, where $T^*\mathbb{P}^1$ is the complex two dimension cotangeant bundle on complex one dimension projective space $\mathbb{P}^1$. In this fibration, which is also equivalent to eq(1.5), the holomorphic variable $\sigma$ parameterizes the base $\mathbb{C}^*$ and the other projective coordinates $(x, y, z, w)$ parameterize the fiber $T^*\mathbb{P}^1$. In this view, conifold is given by a projective hypersurface embedded in $W\mathbb{P}^4(-1,1,-1,1,-1)$.

The power of this way of doing is that it offers a natural method to deal with projective functions $\mathcal{G}(\sigma, x, y, z, w)$ living on conifold and its sub-manifolds. For projective invariant functions $\mathcal{G}(\sigma, x, y, z, w)$ on conifold,

$$\mathcal{G}\left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right) = \mathcal{G}(\sigma, x, y, z, w), \tag{4.13}$$

we have two kinds of expansions: (1) a first expansion given by the usual Laurent development on the base $\mathbb{C}^*$,

$$\mathcal{G}(\sigma, x, y, z, w) = \sum_{n=-\infty}^{\infty} \sigma^n G_n(x, y, z, w), \tag{4.14}$$

with Laurent modes,

$$G_{\mp n}(x, y, z, w) = \frac{1}{2i\pi}\int_{C_0} \frac{d\sigma}{\sigma^{\pm n+1}} \mathcal{G}(\sigma, x, y, z, w), \tag{4.15}$$

where $C_0$ is a contour integral surrounding the pole singularity $\sigma = 0$. (2) Viewed from the fiber sub-manifold, the modes $G_{\mp n}(x, y, z, w)$ are projective functions on $T^*\mathbb{P}^1$ with an integer degree obeying the homogeneity property,

$$G_{\pm n}\left(\lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right) = \lambda^{\pm n} G_{\pm n}(x, y, z, w). \tag{4.16}$$

These are just spin $(|n|+1, 0)$ and spin $(0, |n|+1)$ representations of the $SL(2, C)$ isometry group. So they may be expanded in a harmonic series involving $SL(2)$ homogeneous polynomials. For later use, let us give some details here below; for a complete harmonic space analysis see [29].

## 4.2 Super QFT$_4$ realization

As we will not have the occasion to discuss in details the super QFT$_4$ realization of this projective symmetry in forthcoming sections, let us take this opportunity to fix the ideas by giving a superfield theoretical model realizing this conifold projective geometry. The simplest model one can imagine is given by a $U_{gauge}(1)$ supersymmetric gauge theory with a global $SU(2)$ R-symmetry. The superfield degrees of freedom involved in this gauge theory are reported on following table. They carry quantum numbers $(q,r)$ indicating representations.of the $U_{gauge}(1) \times SU_{global}(2)$ symmetry.

| 4D $\mathcal{N}=1$ Superfields | $(q,r)$ Representations |
|---|---|
| $V = -\theta\sigma^\mu\overline{\theta}A_\mu - i\overline{\theta}^2\theta\lambda + i\theta^2\overline{\theta\lambda} + \frac{1}{2}\theta^2\overline{\theta}^2 D$ | $(0,1)$ real gauge multiplet |
| $\Phi = \phi + \theta\psi + \theta^2\mathrm{F}$, | $(0,1)$ adjoint matter multiplet |
| $Q_{+\alpha} = q_\alpha + \theta\chi_\alpha + \theta^2 F_\alpha$ | $(1,2)$ fundamental matter |
| $P_{-\beta} = p_\beta + \theta\varphi_\beta + \theta^2 G_\beta$ | $\left(-1,\overline{2}\right)$ fundamental matter |
| $\Sigma_\pm = \sigma_\pm + \theta\eta_\pm + \theta^2 L_\pm$ | $(\pm 1,1)$ fundamental matter |
| $\Psi = \gamma_0 + \theta\tau_0 + \theta^2 G$ | $(0,1)$ auxiliary superfield |

(4.17)

where the indices 0, + and − carried by $\Sigma_\pm$ singlets and the $SU(2)$ superfield doublets $Q_{+\alpha}$ and $P_{-\alpha}$ refer respectively to the charges of the $U(1)$ gauge group. For convenience, it is interesting to split the superfields $Q_{+\alpha}$ and $P_{-\alpha}$ as follows,

$$\begin{array}{lll} \text{hypermultiplet } \#\ 1 & : & Q_{+\alpha} = (Q_{+,+} \ \ , \ \ Q_{+,-}) = (X_+ \ \ , \ \ Z_+)\,, \\ \text{hypermultiplet } \#\ 2 & : & P_{-\alpha} = (P_{-,+} \ \ , \ \ P_{-,-}) = (Y_- \ \ , \ \ W_-)\,. \end{array} \tag{4.18}$$

Then introduce the following neutral chiral superfield doublets to be considered later,

$$\begin{array}{lll} \text{hypermultiplet } \#\ 3 & : & H_{0\alpha} = (H_{0,+} \ \ , \ \ Q_{0,-}) = (X_1 \ \ , \ \ X_2) \\ \text{hypermultiplet } \#\ 4 & : & K_{0\alpha} = (K_{0,+} \ \ , \ \ K_{0,-}) = (Y_1 \ \ , \ \ Y_2)\,. \end{array} \tag{4.19}$$

The superspace lagrangian density $\mathcal{L} = \mathcal{L}(T^*\mathbb{S}^3)$ describing the dynamics of these superfields and preserving $U_{gauge}(1) \times SU_{global}(2)$ symmetry may be split as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + (\mathcal{L}_3 + hc)\,. \tag{4.20}$$

The first term reads as,

$$\mathcal{L}_1 = \mathcal{L}_g(V) + \mathcal{L}_{ad}(\Phi) - 2\zeta\int d^4\theta V - \left(\mu\int d^2\theta\Phi + hc\right), \tag{4.21}$$

where $\mathcal{L}_g(V)$ and $\mathcal{L}_{ad}(\Phi)$ stand respectively for the usual gauge covariant lagrangian densities of the $U(1)$ vector multiplet and adjoint matter superfield and where the parameters $\zeta$ and $\mu$ are the usual Fayet Iliopoulos coupling constants. The second term is

given by the usual gauge covariant kinetic terms,

$$\mathcal{L}_2 = \int d^4\theta \sum_{\alpha=1}^{2} \left( \overline{(Q_{+\alpha})} e^{2V} Q_{+\alpha} + \overline{(P_{-\alpha})} e^{-2V} P_{-\alpha} \right) + \int d^4\theta \overline{(\Sigma_\pm)} e^{-2V} \Sigma_\pm . \tag{4.22}$$

The third term $\mathcal{L}_3 = \mathcal{L}_3 \left( \Psi, \Phi, Q_{+\alpha}, P_{-\beta}, \Sigma_\pm \right)$ deals with the chiral and antichiral superpotential. The chiral sector factor reads as follows,

$$\begin{aligned} \mathcal{L}_3 \quad &= \quad \int d^2\theta \left( g_0 \Psi \left( \Sigma_+ \Sigma_- - 1 \right) \right) \\ &\quad + \int d^2\theta \left( g_1 \Phi Q_{+\alpha} P_{-\beta} \varepsilon^{\alpha\beta} - g_2 \Phi \Sigma_+ \Sigma \right) \\ &\quad + \int d^2\theta \mathrm{W} \left( Q, P, \Sigma \right), \end{aligned} \tag{4.23}$$

where the $g_i$'s are coupling constants. Note that eliminating the auxiliary superfields $\Psi$ and $\Phi$ through their holomorphic eqs of motion, one gets,

$$\begin{aligned} \Sigma_+ \Sigma_- \quad &= \quad 1, \\ g_1 Q_{+\alpha} P_{-\beta} \varepsilon^{\alpha\beta} \quad &= \quad \mu + g_2 \Sigma_+ \Sigma . \end{aligned} \tag{4.24}$$

We also recall the following useful relations,

$$\begin{aligned} Q_{+\alpha} P_{-\beta} \varepsilon^{\alpha\beta} \quad &= \quad X_+ Y_- - Z_+ W_-, \\ H_{0\alpha} K_{0\beta} \varepsilon^{\alpha\beta} \quad &= \quad X_1 Y_2 - X_2 Y_1, \\ Q_{+\alpha} Q_{+\beta} \varepsilon^{\alpha\beta} \quad &= \quad P_{-\alpha} P_{-\beta} \varepsilon^{\alpha\beta} = H_{0\alpha} H_{0\beta} \varepsilon^{\alpha\beta} = 0. \end{aligned} \tag{4.25}$$

Substituting $\Sigma_+ \Sigma_- = 1$ in the first relation and shifting $\mu \to g_1 \left( \mu - g_2 \right)$, we discover $Q_{+\alpha} P_{-\beta} \varepsilon^{\alpha\beta} = \mu$. Setting $P_{-\alpha} = K_{0\alpha} \Sigma_-$ and $H_{0\alpha} = \Sigma_- Q_{+\alpha}$, we end with,

$$H_{0\alpha} K_{0\beta} \varepsilon^{\alpha\beta} = \mu, \tag{4.26}$$

which, by help of the identity (4.25) is nothing but the conifold eq $X_1 Y_2 - X_2 Y_1 = \mu$ in superfield language.

# 5 Diffeomorphisms

In the old coordinate system $\{x_i, y_j\}$ where conifold is seen as hypersurface $\varepsilon^{ij} x_i y_j = \mu$ embedded in $\mathbb{C}^4$ eqs(3.1-3.3), conifold isometries are given by the general coordinate transformations

$$\begin{aligned} x'_i \quad &= \quad x'_i \left( x, y \right), \\ y'_i \quad &= \quad y'_i \left( x, y \right), \end{aligned} \tag{5.1}$$

leaving $\varepsilon^{ij}x_iy_j$ invariant. Since $x_i$ and $y_j$ are rotated under $SL(2)$, it is not difficult to see that these general coordinate transformations should be as,

$$
\begin{aligned}
x'_i &= y_k \Upsilon_i^k, \qquad \Upsilon_i^k = \Upsilon_i^k(x, y), \\
y_j &= \Gamma_j^l x_l \qquad \Gamma_j^l = \Gamma_j^l(x, y),
\end{aligned}
\tag{5.2}
$$

where the local matrices $\Upsilon_i^k$ and $\Gamma_j^l$ are constrained as

$$
\varepsilon^{ij}\Upsilon_i^k \Gamma_j^l = 1,
\tag{5.3}
$$

showing $\Gamma_j^l$ is the inverse of $\Upsilon_i^k$ and $\det \Upsilon = 1$. Global matrices $\Upsilon$ generate then the global $SL(2)$ sub-symmetry of $diff(T^*S^3)$.

In the projective coordinate frame $(\sigma, x, y, z, w)$ of $W\mathbb{P}^4(-1, 1, -1, 1, -1)$ eqs(4.5-4.7), we have a quite similar description, except that now we have more explicited transformation from which we can also read the change concerning the base sub-manifold and fiber. In this frame, conifold diffeomorphisms are given by general coordinate transformations,

$$
\begin{aligned}
\sigma' &= \sigma'(\sigma, x, y, z, w), \\
x' &= x'(\sigma, x, y, z, w), \\
y' &= y'(\sigma, x, y, z, w), \\
z' &= z'(\sigma, x, y, z, w), \\
w' &= w'(\sigma, x, y, z, w),
\end{aligned}
\tag{5.4}
$$

preserving projective symmetry; i.e,

$$
\begin{aligned}
\sigma'\left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right) &= \frac{1}{\lambda}\sigma'(\sigma, x, y, z, w) \\
x'\left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right) &= \lambda x'(\sigma, x, y, z, w) \\
y'\left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right) &= \frac{1}{\lambda}y'(\sigma, x, y, z, w) \\
z'\left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right) &= \lambda z'(\sigma, x, y, z, w) \\
w'\left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right) &= \frac{1}{\lambda}w'(\sigma, x, y, z, w).
\end{aligned}
\tag{5.5}
$$

Restricting the variable change $(x', y', z', w')$ to the special case $x' = x'(x, y, z, w)$ and so on, gives diffeomorphisms of $T^*\mathbb{P}^1$. Restriction to the base $\sigma' = \sigma'(\sigma)$ gives $\mathbb{C}^*$ diffeomorphisms. Note that fixing $\sigma' = \sigma$, the above changes reduce to general coordinate transformations on $T^*\mathbb{P}^1$ fiber. The same observations are valid for the $\mathbb{S}^3$, $\mathbb{S}^2$ and $\mathbb{S}^1$ real

slices. The general coordinates transformations (5.4-5.5) translate into the corresponding two component projective isospinor formalism as follows,

$$\begin{aligned} u_i' &= u_i'(\sigma, u_j, v_j), \\ v_i' &= v_i'(\sigma, u_j, v_j), \end{aligned} \tag{5.6}$$

with

$$\begin{aligned} u_i'\left(\frac{1}{\lambda}\sigma, \lambda u_j, \frac{1}{\lambda}v_j\right) &= \lambda u_i'(\sigma, u_j, v_j), \\ v_i'\left(\frac{1}{\lambda}\sigma, \lambda u_j, \frac{1}{\lambda}v_j\right) &= \frac{1}{\lambda}v_i'(\sigma, u_j, v_j). \end{aligned} \tag{5.7}$$

Notice that here $u_j = (x, z)$ and $v_j = (y, w)$ are projective coordinates; they should not be confused with those given by eqs(5.1).

The charge operator of the projective symmetry on $\mathbb{WP}^4(-1, 1, -1, 1, -1)$ namely,

$$\nabla_0 = \left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\right) - \left(y\frac{\partial}{\partial y} + w\frac{\partial}{\partial w}\right) - \sigma\frac{\partial}{\partial \sigma}, \tag{5.8}$$

splits into two commuting parts as

$$\nabla_0 = 2D_0 - T_0 \tag{5.9}$$

with contribution $T_0$ coming from the base,

$$T_0 = \nabla_0|_{\mathbb{C}^*}, \tag{5.10}$$

and a second one $D_0$ coming from the fiber,

$$2D_0 = \nabla_0|_{T^*\mathbb{P}^1}. \tag{5.11}$$

The charge operator $T_0$, which counts the projective charges of sections along the $\mathbb{C}^*$ base, reads as,

$$T_0 = \sigma\frac{\partial}{\partial \sigma}, \tag{5.12}$$

and has integer eigenvalues $n$ and eigen-functions given by the usual Laurent monomials $f_n \sim \sigma^n$. The $D_0$ charge operator deals with the counting of the Cartan Weyl charges on the $T^*\mathbb{P}^1$ base; it reads then as,

$$2D_0 = \left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\right) - \left(y\frac{\partial}{\partial y} + w\frac{\partial}{\partial w}\right). \tag{5.13}$$

It has integer eigenvalues and eigen-functions given by harmonic functions on $T^*\mathbb{P}^1$. In this setting, projective invariance of functions $G = G(T^*S^3)$ living on $T^*S^3$, is solved as,

$$\nabla_0 G(\sigma, x, y, z, w) = 0, \tag{5.14}$$

implying in turns,

$$2D_0G = T_0G. \tag{5.15}$$

This identity means that projective charges on fiber and base sub-manifolds cancel themselves exactly. Notice that in general, projective covariance on conifold requires,

$$eignvalue\,(2D_0) - eignvalue\,(T_0) \quad \in \quad \mathbb{Z}. \tag{5.16}$$

On the cotangeant bundle $T^*\mathbb{P}^1$ described by the projective invariant eq $xy - zw = \mu$ with $x \equiv \lambda x$, $y \equiv \frac{1}{\lambda}y$, $z \equiv \lambda z$ and $w \equiv \frac{1}{\lambda}w$ but no $\sigma$ dependence, it happens that $D_0$ is in fact one of a set of three operators namely $D_0$ and $D_\pm$. These operators generate the $SL\,(2,\mathbb{C})$ global sub-group of conifold diffeomorphism isometries. The commutation relations of $D_0$ and $D_\pm$ are given by similar relations to those of $SU\,(2,\mathbb{C})$ except here we have no hermiticity conditions,

$$\begin{aligned} [D_+, D_-] &= 2D_0, \\ [2D_0, D_\pm] &= \pm 2D_\pm. \end{aligned} \tag{5.17}$$

Together with the expression of $2D_0$ given above eq(5.13), the $D_\pm$ operators realizing the above $SL\,(2,\mathbb{C})$ brackets read as,

$$\begin{aligned} D_+ &= x\frac{\partial}{\partial w} - z\frac{\partial}{\partial y}, \\ D_- &= w\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}. \end{aligned} \tag{5.18}$$

Under this realization of $SL\,(2,\mathbb{C})$ global isometry, one recovers the results described above. In particular we recover that the projective variables $u = (x,z)$ and $v = (y,w)$ carry respectively the projective charges $(+,+)$ and $(-,-)$ and moreover transform as $SL\,(2,\mathbb{C})$ isodoublets as shown below,

$$[\nabla_0, (x,z)] = (x,z)\,, \qquad [\nabla_0, (y,w)] = (-y,-z)\,, \tag{5.19}$$

and

$$\begin{aligned} [\nabla_+, (x,z)] &= (0,0) \\ [\nabla_-, (x,z)] &= (w,-y)\,, \\ \left[\nabla_-^2, (x,z)\right] &= (0,0)\,. \end{aligned} \tag{5.20}$$

Similar relations for $y$ and $w$ may be also written down; in particular we have $[\nabla_-, (y,w)] = (0,0)$. To complete the picture let us make three comments concerning diffeomorphism isometries of the conifold.

First note that in the projective frame, there is a one to one correspondence between base and fiber objects,

| $T^*\mathbb{P}^1$ fiber | $\leftrightarrow$ | $\mathbb{C}^*$ base | $\leftrightarrow$ | Conifold |
|---|---|---|---|---|
| $G_n = G_n(x,y,z,w)$ | | $f_n = \sigma^n$ | | $G(\sigma,x,y,z,w) = \sum \sigma^n G_n$ |
| $D_+ = x\frac{\partial}{\partial w} - z\frac{\partial}{\partial y}$ | | $T_+ = \frac{1}{\sqrt{2}}\frac{\partial}{\sigma\partial\sigma}$ | | $\nabla_+ = D_+ - T_+$ |
| $2D_0 = \frac{x\partial}{\partial x} + \frac{z\partial}{\partial z} - \frac{y\partial}{\partial y} - \frac{w\partial}{\partial w}$ | | $T_0 = \sigma\frac{\partial}{\partial\sigma}$ | | $\nabla_0 = 2D_0 - T_0$ |
| $D_- = w\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}$ | | $T_- = -\frac{\sigma^3\partial}{\sqrt{2}\partial\sigma}$ | | $\nabla_- = D_- - T_-$ |
| $2D_0G_n = nG_n$ | | $T_0 f_n = n f_n$ | | $\nabla_0 G = 0$ |

, (5.21)

where $n \in Z$. Second, like $D_0$ and $D_\pm$, the generators $(-T_0)$ and $(-T_\pm)$ obey an $SL(2,C)$ algebra. In addition to eq(5.12), the realization of the generators $T_\pm$ is given by,

$$T_+ = \frac{1}{\sqrt{2}}\frac{\partial}{\sigma\partial\sigma}, \qquad T_- = -\frac{1}{\sqrt{2}}\frac{\sigma^3\partial}{\partial\sigma}. \tag{5.22}$$

It is dictated by the projective transformation $\sigma \to \frac{1}{\lambda}\sigma$ acting on $T_\pm$ as $\lambda^{\pm 2}T_\pm$ that is in same manner as does the change $(x,y,z,w) \to \left(\lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right)$ on the $D_\pm$ generators of the $T^*\mathbb{P}^1$ base. It follows also from the representation

$$T_+ = \frac{1}{\sqrt{2}}\sigma_+\frac{\partial}{\partial\sigma_-}, \qquad T_- = \frac{1}{\sqrt{2}}\sigma_-\frac{\partial}{\partial\sigma_+}, \qquad T_0 = \frac{1}{2}\left(\sigma_+\frac{\partial}{\partial\sigma_+} - \sigma_-\frac{\partial}{\partial\sigma_-}\right), \tag{5.23}$$

and substituting $\sigma_+\sigma_- = 1$. Finally under global projective symmetry, the $SL(2)$ generators of conifold scale in same manner as for $D_q$ and $T_q$ parts namely,

$$\nabla_q \to \lambda^{2q}\nabla_q, \tag{5.24}$$

with $q = 0, \pm 1$.

## 5.1 Conifold isometries

First note that, along with the particular global projective symmetry described above (4.7), conifold has an infinite set of diffeomorphism symmetries sitting in the group $Diff(T^*\mathbb{S}^3)$. Roughly speaking, and as far as the $T^*\mathbb{P}^1$ fiber is concerned, there are three main subsets which read in the $T^*\mathbb{P}^1$ projective isospinor formalism as,

$$\begin{pmatrix} u \\ v \end{pmatrix} \to M \begin{pmatrix} u \\ v \end{pmatrix}, \tag{5.25}$$

with matrix $M$ as follows,

$$M_0 = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & \Upsilon \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 \\ \Gamma & 1 \end{pmatrix}. \tag{5.26}$$

where $\Lambda \neq 0$, $\Upsilon$ and $\Gamma$ are arbitrary functions on conifold. To complete the picture, one should also add the general coordinate transformation of the base $\sigma' = \sigma'(\sigma, u, v)$; it will be implemented later. A particular subset of $u$ and $v$ general coordinate transformations is that given by the holomorphic change $M_1$ mapping half of the $T^*\mathbb{P}^1$ projective coordinate, say $x$ and $z$, as

$$\begin{aligned} x &\rightarrow x'(\sigma, x, y, z, w) = x + \varepsilon(\sigma, x, y, z, w), \\ z &\rightarrow z'(\sigma, x, y, z, w) = z + \epsilon(\sigma, x, y, z, w), \end{aligned} \tag{5.27}$$

and fixing the other half since leaving the variables $y$ and $w$ unchanged,

$$y \rightarrow y' = y, \qquad\qquad w \rightarrow w' = w. \tag{5.28}$$

In the projective coordinate frame we are working with here, explicit general coordinates transformations that leave invariant conifold hypersurface $xy - zw = \mu$ read as,

$$\begin{aligned} x &\rightarrow x' = x + \Upsilon w, \\ z &\rightarrow z' = z + \Upsilon y, \\ y &\rightarrow y' = y, \qquad w \rightarrow w' = w. \end{aligned} \tag{5.29}$$

If focusing on the $T^*\mathbb{P}^1$ coordinates where projective symmetry should be imposed, the general coordinates parameter $\Upsilon = \Upsilon(\sigma, x, y, z, w)$ is an arbitrary degree two homogeneous function on conifold,

$$\Upsilon\left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right) = \lambda^2 \Upsilon(\sigma, x, y, z, w). \tag{5.30}$$

This is because of the opposite projective charges of $(x, z)$ and $(y, w)$. According to the analysis of previous section, this function expands in a Laurent series as follows,

$$\Upsilon(\sigma, x, y, z, w) = \sum_{n=-\infty}^{\infty} \sigma^n \Upsilon_{n+2}(x, y, z, w), \tag{5.31}$$

with

$$\Upsilon_{n+2} = \oint \frac{d\sigma}{2i\pi\sigma^{n+1}} \Upsilon, \tag{5.32}$$

being the Laurent modes and at same time are functions living on $T^*\mathbb{P}^1$. Recall that $x$ and $z$ have projective degree one and $\sigma$, $y$ and $w$ have a degree minus one. The constraint eq(5.30), which reads also as

$$[2\nabla_0, \Upsilon] = 2\Upsilon, \tag{5.33}$$

or equivalently by using the splitting $2\nabla_0 = 2D_0 - T_0$,

$$\begin{aligned} [2D_0, \sigma^n \Upsilon_{n+2}] &= (n+2)\Upsilon_{n+2}, \\ [T_0, \sigma^n \Upsilon_{n+2}] &= n\Upsilon_{n+2}, \end{aligned} \tag{5.34}$$

has infinitely many solutions for $\Upsilon_{n+2}$ classified by $SL(2,\mathbb{C})$ spins $(s_1, s_2)$. The two simplest examples read respectively as

$$\Upsilon = \frac{1}{\sigma^2}, \qquad D_{0,\pm}\left(\frac{1}{\sigma^2}\right) = 0, \tag{5.35}$$

living on base; i.e no dependence on fiber coordinate variables, and

$$\Upsilon = \frac{ax+bz}{\sigma}, \qquad 2D_0\left(\frac{ax+bz}{\sigma}\right) = -T_0\left(\frac{ax+bz}{\sigma}\right) = \frac{ax+bz}{\sigma}, \tag{5.36}$$

with a foot in the base and the other in the fiber. The next example coming after is given by the following isotriplet representation $(1,0)$ living in the base,

$$\Upsilon \equiv \Upsilon_2 = \left(ax^2 + bxz + cz^2\right), \qquad T_{0,\pm}\left(\Upsilon_2\right) = 0, \tag{5.37}$$

with $a$, $b$ and $c$ are complex parameters. Form eq(5.13), one can easily check that the property $[2D_0 - T_0, \Upsilon] = 2\Upsilon$ is fulfilled; and by using eq(5.18), one finds that it satisfies moreover $[D_+, \Upsilon_2] = 0$ showing that $\Upsilon_2$ is indeed a highest weight state of spin $(1,0)$.

Together with the general coordinate transformations (5.27), we have also a mirror set of diffeomorphisms fixing the isodoublet $(x,z)$, that is $x \to x' = x$, $z \to z' = z$, but changing the second isodoublet $(y,w)$ as follows,

$$\begin{aligned} y &\rightarrow y' = y + \Gamma z, \\ w &\rightarrow w' = w + \Gamma x. \end{aligned} \tag{5.38}$$

Here also, the diffeomorphism group parameter $\Gamma = \Gamma(\sigma, x, y, z, w)$ is given by homogeneous function living on conifold and has a degree $(-2)$,

$$\Gamma\left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w\right) = \lambda^{-2}\Gamma(\sigma, x, y, z, w). \tag{5.39}$$

Like before, this holomorphic function expands in a Laurent series as $\Gamma(\sigma, x, y, z, w) = \sum_{n=-\infty}^{\infty} \sigma^n \Gamma_{n-2}(x, y, z, w)$. In the $SL(2,\mathbb{C})$ differential operator language, the condition (5.39) maps to,

$$[2D_0, \Gamma_{n-2}] = (n-2)\,\Gamma_{n-2}, \tag{5.40}$$

and has infinitely many solutions. The simplest three solutions read respectively as $\Gamma = \sigma^2$ living on base, $\Gamma = \sigma(ay + bw)$ with a foot in fiber and the other in base and the third one is given by the $(0,1)$ isotriplet representation,

$$\Gamma = \left(ay^2 + byw + cz^2\right), \tag{5.41}$$

with $a$, $b$ and $c$ some arbitrary group parameters. It lives in the $T^*\mathbb{P}^1$ fiber sub-manifold.

## 5.2 Local $\mathbb{C}^*$ symmetry

Here we complete the previous analysis by considering the study of local projective symmetry (4.7). This concerns the abelian gauge sub-symmetry obtained by setting $\Upsilon = 0$, $\Gamma = 0$ and $\digamma = 0$ in the following typical general coordinate transformations. Recall that the general coordinates transformations in u-sector reads as,

$$\begin{aligned}
x &\rightarrow x' = \Lambda x + \Lambda \Upsilon w, \\
z &\rightarrow z' = \Lambda z + \Lambda \Upsilon y, \\
y &\rightarrow y' = \frac{1}{\Lambda} y, \qquad w \rightarrow w' = \frac{1}{\Lambda} w, \\
\sigma &\rightarrow \sigma' = \frac{1}{\Lambda}(\sigma + \digamma),
\end{aligned} \tag{5.42}$$

with gauge parameters $\Lambda = \Lambda(\sigma, x, y, z, w)$, $\Upsilon = \Upsilon(\sigma, x, y, z, w)$ and $\digamma = \digamma(\sigma, x, y, z, w)$ while their analog for $v$-sector have the form,

$$\begin{aligned}
x &\rightarrow x' = \Lambda x, \qquad z \rightarrow z' = \Lambda z, \\
\sigma &\rightarrow \sigma' = \frac{1}{\Lambda}(\sigma + \digamma), \\
y &\rightarrow y' = \frac{1}{\Lambda} y + \frac{\Gamma}{\Lambda} z, \\
w &\rightarrow w' = \frac{1}{\Lambda} w + \frac{\Gamma}{\Lambda} x,
\end{aligned} \tag{5.43}$$

As noted previously, since the defining equation of the conifold $\det(YX) = \mu$ involves no coordinate derivatives, the projective change

$$Y' = \mathbf{\Lambda} Y, \qquad X' = X \mathbf{\Lambda}^{-1}, \tag{5.44}$$

with $\mathbf{\Lambda}$ as in eq(4.11), is also valid for a local group parameter

$$\Lambda = \Lambda(\sigma, x, y, z, w), \tag{5.45}$$

living on conifold. To fix the ideas, $\Lambda$ may be thought of as given by,

$$\Lambda(\sigma, x, y, z, w) = \lambda \exp[\eta(\sigma, x, y, z, w)], \tag{5.46}$$

where the non zero complex constant $\lambda$ is as before and where $\eta = \eta(\sigma, x, y, z, w)$ is an arbitrary local projective function. Like before, a class of these functions $\Lambda$ is given by homogeneous a function of degree zero; i.e,

$$\Lambda\left(\frac{1}{\lambda}\sigma, \lambda x, \frac{1}{\lambda} y, \lambda z, \frac{1}{\lambda} w\right) = \Lambda(\sigma, x, y, z, w), \tag{5.47}$$

Conservation of the global projective charge shows that this kind of function $\Lambda$ may be expanded in a Laurent series as follows,

$$\Lambda(\sigma, x, y, z, w) = \sum_{n=-\infty}^{\infty} \sigma^n \Lambda_n, \tag{5.48}$$

where $\Lambda_n = \Lambda_n(x, y, z, w)$ are projective functions of order $n$ living on $T^*\mathbb{P}^1$ and satisfying,

$$[2D_0, \Lambda_n] = n\Lambda_n \tag{5.49}$$

Like before, there are infinitely many solutions classified by $SL(2, \mathbb{C})$ representations. The simplest one is naturally the global constant $\Lambda_0 = \lambda$ of spin $(s_1, s_1) = (0, 0)$ while the two next ones read as

$$\Lambda = \sigma(ax + bz), \tag{5.50}$$

and

$$\Lambda = \frac{ay + bw}{\sigma}, \tag{5.51}$$

and are respectively associated with $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ representations of $SL(2)$ global isometry group. The next solution, which is given by the spin $\left(\frac{1}{2}, \frac{1}{2}\right)$, reads as

$$\Lambda = axw + bzy + c(xy + zw), \tag{5.52}$$

with $a$, $b$ and $c$ are arbitrary complex parameters. As such the global projective eqs(4.11) extends locally as

$$\begin{aligned} x' &= \Lambda x, \qquad z' = \Lambda z, \\ y' &= \frac{1}{\Lambda} y, \qquad w' = \frac{1}{\Lambda} w, \end{aligned} \tag{5.53}$$

and is interpreted as a $\mathbb{C}^*$ gauge symmetry acting on projective functions living on conifold. In next section, we fix our attention on this abelian local symmetry and look for the corresponding underlying gauge theory.

# 6 More on gauging $\mathbb{C}^*$ isometry

To start recall that in the projective transformation (4.7), the parameter $\lambda$ is a non zero global $SL(2, \mathbb{C})$ scalar; that is a non zero complex constant satisfying

$$[D_0, \lambda] = [D_\pm, \lambda] = 0, \qquad [T_0, \lambda] = [T_\pm, \lambda] = 0, \tag{6.1}$$

that is

$$[\nabla_0, \lambda] = [\nabla_\pm, \lambda] = 0. \tag{6.2}$$

Under this global $\mathbb{C}^*$ symmetry, generic field sections $G_n = G_n(x, y, z, w)$ with $n$ charges and their derivatives $D_{0,\pm}G_n$ transform covariantly as in eq(4.16) namely

$$G_n \to \lambda^n G_n, \qquad (D_{0,\pm}G_n) \to \lambda^n (D_{0,\pm}G_n). \tag{6.3}$$

The same transformations are valid for the global $SL(2, \mathbb{C})$ generators $D_{0,\pm}$ which transform then as,

$$D_\pm \to D'_\pm = \lambda^{\pm 2} D_\pm, \qquad 2D_0 \to 2D'_0 = 2D_0. \tag{6.4}$$

Similar relations may be written down for the analogous base quantities, for instance $\sigma^n \rightarrow \lambda^{-n}\sigma^n$ and $T_0 \rightarrow T_0' = T_0$, $T_\pm \rightarrow T_\pm' = \lambda^{\pm 2}T_\pm$. The same is valid for $\nabla_{0,\pm}$, eq(5.24). Extending the global projective symmetry of parameter $\lambda$ to a local one with arbitrary gauge parameter $\Lambda$ living on conifold,

$$[\nabla_{0,\pm}, \Lambda] \neq 0, \tag{6.5}$$

the conifold hypersurface $\{(\sigma, x, y, z, w) \mid xy - zw = \mu; \quad \sigma \in \mathbb{C}^*\}$ remains invariant. But what about field on conifold and their derivatives. To that purpose let us first focus on the field sections $G_n(x, y, z, w)$ on $T^*\mathbb{P}^1$. These fields transform covariantly as $G_n \rightarrow \Lambda^n G_n$; however the derivatives $D_\pm G_n$ and $D_0 G_n$ are no longer covariant since they undergo like,

$$\begin{aligned} D_\pm G_n &= \Lambda^{n\pm2}\left[D_\pm + nD_\pm(\ln\Lambda)\right]G_n, \\ D_0 G_n &= \Lambda^n\left[D_0 + nD_0(\ln\Lambda)\right]G_n. \end{aligned} \tag{6.6}$$

These are typical transformations in gauge theories which rests on the fact that the derivatives $D_{0,\pm}$ are not covariant. Using the explicit expressions of $D_{0,\pm}$ and the local transformations (5.53), we get the following,

$$\begin{aligned} D_+ &= D_+' + 2(D_+\ln\Lambda)D_0' \\ D_- &= D_-' + 2(D_-\ln\Lambda)D_0' \\ D_0 &= D_0' + 2(D_0\ln\Lambda)D_0', \end{aligned} \tag{6.7}$$

where one recognizes $2D_0$ as the generator of $\mathbb{C}^*$ isometry. To restore $\mathbb{C}^*$ gauge covariance, one should introduce the holomorphic gauge fields

$$A_{0,\pm} = A_{0,\pm}(\sigma, x, y, z, w), \tag{6.8}$$

in order to covariantize the derivatives (5.13,5.18) which becomes then,

$$\begin{aligned} \mathcal{D}_\pm &= D_\pm - A_\pm D_0, \\ 2\mathcal{D}_0 &= 2D_0 - 2A_0 D_0, \end{aligned} \tag{6.9}$$

Note that on $T^*\mathbb{P}^1$, we should have $\mathcal{D}_0 = D_0$, that is $A_0 = 0$ while on conifold such constraint equation on the charge has no place. The gauge transformations of $A_\pm$ and $A_0$ are obtained by requiring $\mathcal{D}_\pm G_n$ to transform covariantly

$$\begin{aligned} \mathcal{D}_\pm G_n &= \Lambda^{n\pm2}\mathcal{D}_\pm G_n, \\ \mathcal{D}_0 G_n &= \Lambda^n \mathcal{D}_0 G_n, \end{aligned} \tag{6.10}$$

which imply in turn the following gauge transformation of the gauge fields

$$\begin{aligned} A_\pm G_n &\rightarrow \Lambda^{\pm2}\left[A_\pm + n\mathcal{D}_\pm(\ln\Lambda)\right](\Lambda^n G_n), \\ A_0 G_n &\rightarrow \left[A_0 + n\mathcal{D}_0(\ln\Lambda)\right]\Lambda^n G_n. \end{aligned}$$

They may be directly obtained from eqs(6.9) by solving the constraint eqs $\mathcal{D}'_{\pm} = \Lambda^{\pm 2}\mathcal{D}_{\pm}$ and $\mathcal{D}'_0 = \mathcal{D}_0$. We find,

$$\begin{aligned} A_{\pm} &\rightarrow A'_{\pm} = \Lambda^{\pm 2}\left[A_{\pm} + D_{\pm}(\ln\Lambda)\right], \\ A_0 &\rightarrow A'_0 = A_0 + 2D_0(\ln\Lambda). \end{aligned} \tag{6.11}$$

For the particular case of a global transformation where $\Lambda$ is restricted to $\lambda$; that is $D_{0,\pm}\Lambda = D_{0,\pm}\lambda = 0$, one recovers the usual covariance of $A_{0,\pm}$ as holomorphic global field sections.

These results for the fiber $T^*\mathbb{P}^1$ extend obviously to the $SL(2)$ generators $T_{0,\pm}$ on the base and more generally to $\nabla_{0,\pm}$. On conifold, the previous fiber gauge transformations read as,

$$\begin{aligned} \nabla_{\pm} &= \nabla'_{\pm} + 2(\nabla_{\pm}\ln\Lambda)\nabla'_0 \\ \nabla_0 &= \nabla'_0 + 2(\nabla_0\ln\Lambda)\nabla'_0. \end{aligned} \tag{6.12}$$

The corresponding gauge covariant derivatives are given by,

$$\begin{aligned} \boldsymbol{\nabla}_{\pm} &= \nabla_{\pm} - A_{\pm}\nabla_0, \\ 2\boldsymbol{\nabla}_0 &= 2\nabla_0 - 2A_0\nabla_0, \end{aligned} \tag{6.13}$$

and the gauge transformations of the gauge fields on conifold are as follows,

$$\begin{aligned} A_{\pm} &\rightarrow A'_{\pm} = \Lambda^{\pm 2}\left[A_{\pm} + \nabla_{\pm}(\ln\Lambda)\right], \\ A_0 &\rightarrow A'_0 = A_0 + 2\nabla_0(\ln\Lambda). \end{aligned} \tag{6.14}$$

With these tools at hand, we turn now to derive the correspondence between conifold geometry and non commutative topological holomorphic $SL(2)$ gauge theory.

## 6.1 Deriving the topological gauge constraint eqs

So far we have considered two coordinate systems to deal with conifold geometry, an old free complex coordinate frame $\{x_i, y_j\}$ and a projective one $\{\sigma, x, y, z, w\}$. In the old coordinate system, conifold is seen as hypersurface $H_0$ embedded in $\mathbb{C}^4$; its defining eq $\varepsilon^{ij}x_iy_j = \mu$ is invariant under the general coordinate change $x'_i = y_k\Upsilon_i^k$, $y_j = (\Upsilon^{-1})_j^l x_l$. Moreover, the generators of the global $SL(2)$ isometry read as follows:

$$\begin{aligned} \nabla_+ &= \frac{1}{\sqrt{2}}\sum_i x_i\frac{\partial}{\partial y_i}, \\ \nabla_- &= \frac{1}{\sqrt{2}}\sum_i y_i\frac{\partial}{\partial x_i}, \\ \nabla_0 &= \frac{1}{2}\sum_i\left(x_i\frac{\partial}{\partial x_i} - y_i\frac{\partial}{\partial y_i}\right). \end{aligned} \tag{6.15}$$

They satisfy the usual commutation relations

$$[\nabla_{+}, \nabla_{-}] = 2\nabla_{0}, \qquad [2\nabla_{0}, \nabla_{\pm}] = \pm 2\nabla_{\pm}. \tag{6.16}$$

Acting on the conifold hypersurface $x_1 y_2 - x_2 y_1 = \mu$ by these operators, one discovers, as expected, the following relations,

$$[\nabla_{+}, H] = [\nabla_{-}, H] = [\nabla_{0}, H] = 0. \tag{6.17}$$

Functions $\mathcal{F}_R\left(T^* S^3\right)$ on conifold hypersurface $H_0$ are functions of the complex coordinates $x_i$ and $y_j$ with the restriction $\varepsilon^{ij} x_i y_j = \mu$ and transforming in representations $R$ of $SL\left(2\right)$ isometry group. Under the change $x_i' = y_k \Upsilon_i^k$, $y_j = \left(\Upsilon^{-1}\right)_j^l x_{il}$, these functions $\mathcal{F}_R\left(T^* S^3\right)$ transform covariantly. For the derivatives $\nabla_{0,\pm}\mathcal{F}_R$ to transform covariantly, one has to introduce gauge connexions,

$$\boldsymbol{\nabla}_q = \nabla_q - A_q, \qquad q = 0, \pm 1, \tag{6.18}$$

where the gauge fields $A_q = \sum_{p=0,\pm 1} B_q^p \nabla_p$ are vector fields valued in $sl\left(2\right)$ algebra. The $A_{0,\pm}$ gauge fields are not all of them independent; their relations are obtained by requiring that gauge covariant derivatives $\boldsymbol{\nabla}_{0,\pm}$ satisfy as well an $sl\left(2\right)$ algebra,

$$\begin{aligned} [\boldsymbol{\nabla}_{+}, \boldsymbol{\nabla}_{-}] &= 2\boldsymbol{\nabla}_{0}, \\ [2\boldsymbol{\nabla}_{0}, \boldsymbol{\nabla}_{+}] &= 2\boldsymbol{\nabla}_{+}, \\ [2\boldsymbol{\nabla}_{0}, \boldsymbol{\nabla}_{-}] &= -2\boldsymbol{\nabla}_{-}. \end{aligned} \tag{6.19}$$

Notice that we have three gauge field components satisfying three constraint eqs. This property signs the topological feature of $sl\left(2\right)$ gauge theory on conifold. This description is also valid for in the projective coordinate system $\{\sigma, x, y, z, w\}$ where conifold is seen as a projective complex three surface embedded in $WP^4\left(-1, 1, -1, 1, -1\right)$. In this case, the previous generators $\boldsymbol{\nabla}_{0,\pm}$ of the global $SL\left(2\right)$ split as;

$$\boldsymbol{\nabla}_{\pm} = \mathcal{D}_{\pm} - \mathcal{T}_{\pm}, \qquad \boldsymbol{\nabla}_{0} = 2\mathcal{D}_{0} - \mathcal{T}_{0}, \tag{6.20}$$

$\mathcal{T}_{0,\pm}$ for the base $\mathbb{C}^*$ and $\mathcal{D}_{0,\pm}$ for fiber $T^*\mathbb{P}^1$. Notice that that the relations (6.19) may be put into a condensed form by help of the completely antisymmetric invariant three dimensional tensor $\varepsilon_{pqr}$ with the usual cyclic property $\varepsilon_{+-0} = \varepsilon_{-0+} = \varepsilon_{0+-} = 1$, one can put above eqs as follows,

$$[\boldsymbol{\nabla}_{p}, \boldsymbol{\nabla}_{q}] = \varepsilon_{pqr}\boldsymbol{\nabla}_{-r}, \tag{6.21}$$

where we have renamed $2\boldsymbol{\nabla}_0$ and $\boldsymbol{\nabla}_{\pm}$ as $\boldsymbol{\nabla}_0'$ and $\sqrt{\frac{1}{2}}\boldsymbol{\nabla}_{\pm}'$ respectively; then dropped out the upper index prime. By using the inverse tensor $\varepsilon^{rpq}$ with the properties $\varepsilon^{rpq}\varepsilon_{rqp} = 2\delta_s^r$ and $\varepsilon^{rpq}\varepsilon_{rqp} = 6$, we can rewrite previous relation as follows,

$$\varepsilon^{rpq}\left[\boldsymbol{\nabla}_{p}, \boldsymbol{\nabla}_{q}\right] = -\frac{1}{2}\boldsymbol{\nabla}_{-r} \tag{6.22}$$

By substituting $\boldsymbol{\nabla}_{\pm} = \nabla_{\pm} - A_{\pm}\nabla_0$ and $\boldsymbol{\nabla}_0 = \nabla_0 - A_0\nabla_0$ back into the constraint eqs(6.21), we get the following,

$$\begin{aligned}
(\nabla_+ A_- - \nabla_- A_+) + (A_+\nabla_0 A_- - A_-\nabla_0 A_+) &= A_0, \\
(\nabla_- A_0 - \nabla_0 A_-) + (A_-\nabla_0 A_0 - A_0\nabla_0 A_-) &= A_-, \\
(\nabla_0 A_+ - \nabla_+ A_0) + (A_0\nabla_0 A_+ - A_+\nabla_0 A_0) &= A_+,
\end{aligned} \tag{6.23}$$

which read by help of gauge covariant derivatives $\boldsymbol{\nabla}_p$ in a condensed form as,

$$\varepsilon_p^{rpq}\boldsymbol{\nabla} A_q + A_{-r} = 0. \tag{6.24}$$

Observe in passing that these constraint eqs are gauge invariant. Indeed under gauge transformations $\delta A_q = \boldsymbol{\nabla}_q(\ln\Lambda)$, the quantity $\varepsilon^{rpq}(\boldsymbol{\nabla}_p A_q)$ varies as $\varepsilon^{rpq}[\boldsymbol{\nabla}_p\boldsymbol{\nabla}_q(\ln\Lambda)]$ which by help of (6.19) we get $\frac{1}{2}\varepsilon^{rpq}\varepsilon_{pq-s}\boldsymbol{\nabla}_s = -\delta A_{-r}$. Note finally that at first sight these constraint eqs (6.23) and analogs look a little bit unusual. While we are dealing with an abelian gauge theory, our constraint eqs have generated non linear terms. This is not a contradiction; it is just a signal of the non commutative behavior of the topological $\mathbb{C}^*$ gauge theory.

## 6.2 The holomorphic topological action

To get the complex holomorphic gauge field action $S_{T^*\mathbb{S}^3} = S_{T^*\mathbb{S}^3}[A_-, A_+, A_0]$ describing the underlying NC topological holomorphic gauge theory, one should think about the previous constraint relations (6.23) as field theoretic equation of motion following from the action principle

$$\frac{\delta S_{T^*\mathbb{S}^3}}{\delta A_r} = 0, \qquad r = 0, +, -. \tag{6.25}$$

To solve this equation, it interesting to first express the action $S_{T^*\mathbb{S}^3}$ as the integral over a holomorphic integral lagrangian like density $\mathcal{L}_{T^*\mathbb{S}^3}(A)$ as,

$$S_{T^*\mathbb{S}^3} = \frac{1}{\lambda}\int_{T^*\mathbb{S}^3} d^3v\mathcal{L}_{T^*\mathbb{S}^3}(A), \tag{6.26}$$

where $\lambda$ is the gauge coupling constant and where the three form $d^3v$ stands for the conifold holomorphic volume form which, in the projective coordinates $(\sigma, x, y, z, w)$, splits as the invariant volume 1-form of the $\mathbb{C}^*$ base times the invariant volume 2-form of the fiber $T^*\mathbb{P}^1$. Projective and $SL(2)$ invariances lead to the following local volume form,

$$d^3v = \frac{d\sigma}{2i\pi\sigma} \times (dx \times dy - dz \times dw) \tag{6.27}$$

Then equating $\frac{\partial \mathcal{L}_{T^*\mathbb{S}^3}}{\delta A_r}$ with $(\varepsilon^{rpq}\boldsymbol{\nabla}_p A_q + A_{-r})$ eq(6.24), which for convenience we rewrite it as

$$\frac{\partial \mathcal{L}_{T^*\mathbb{S}^3}}{\delta A_r} = \varepsilon^{rpq}\left[\boldsymbol{\nabla}_p A_q - \frac{1}{2}\varepsilon_{pqs}A_{-s}\right], \tag{6.28}$$

and integrating with respect to $A_r$, we obtain the following holomorphic field Lagrangian density $\mathcal{L}(A)$,

$$\mathcal{L}_{T^*\mathbb{S}^3} = \frac{1}{2}\varepsilon^{rpq}\left[A_r \boldsymbol{\nabla}_P A_q - \frac{1}{2}\varepsilon_{pqs} A_r A_{-s}\right]. \tag{6.29}$$

Gauge invariance follows naturally from covariance and the complete antisymmetry of $\varepsilon^{rpq}$ tensor. Note that while the first term in bracket is the usual gauge term for abelian gauge theories, the second one is typical for non commutative geometry.

On the real slice of the conifold, the initial $SL(2,\mathbb{C})$ symmetry reduces to $SU(2,\mathbb{C})$ and $\mathbb{C}^*$ to $U(1)$ invariance, in particular we have the following generators

$$\begin{aligned}
\boldsymbol{\nabla}_+|_{S^3} &= \frac{1}{\sqrt{2}}\left(x\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial \overline{x}}\right), \\
\boldsymbol{\nabla}_-|_{S^3} &= \frac{1}{\sqrt{2}}\left(\overline{z}\frac{\partial}{\partial x} - \overline{x}\frac{\partial}{\partial z}\right), \\
\boldsymbol{\nabla}_0|_{S^3} &= \frac{1}{2}\left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\right) - \left(\overline{x}\frac{\partial}{\partial \overline{x}} + \overline{z}\frac{\partial}{\partial \overline{z}}\right).
\end{aligned} \tag{6.30}$$

Similarly, the previous complex holomorphic gauge fields $A_\pm$ and $A_0$ reduce to $C_\pm$ and $C_0$; and obey the following reality conditions,

$$\begin{aligned}
(C_\pm)^\dagger &= C_\mp, \qquad (C_0)^\dagger = C_0, \\
\boldsymbol{\nabla}_{0,\pm} &= \nabla_{0,\pm} - C_{0,\pm},
\end{aligned} \tag{6.31}$$

while the constraint eqs read in the same manner as before,

$$\begin{aligned}
[\boldsymbol{\nabla}_+, \boldsymbol{\nabla}_-] &= \boldsymbol{\nabla}_0, \\
[\boldsymbol{\nabla}_0, \boldsymbol{\nabla}_+] &= \boldsymbol{\nabla}_+, \\
[\boldsymbol{\nabla}_0, \boldsymbol{\nabla}_-] &= -\boldsymbol{\nabla}_-.
\end{aligned} \tag{6.32}$$

It looks like $sl(2)$ relations, except that now we have moreover the reality conditions,

$$(\boldsymbol{\nabla}_\pm)^\dagger = \boldsymbol{\nabla}_\mp, \qquad (\boldsymbol{\nabla}_0)^\dagger = \boldsymbol{\nabla}_0. \tag{6.33}$$

Following the same steps, we end with the non commutative topological Chern-Simons gauge theory

$$S_{\mathbb{S}^3}[C] = \frac{1}{\lambda}\int_{\mathbb{S}^3} \frac{1}{2}\varepsilon^{rpq}\left[C_r \boldsymbol{\nabla}_p C_q - \frac{1}{2}\varepsilon_{pqs} C_r C_{-s}\right], \tag{6.34}$$

Gauge invariance follows in a similar way as for the complex holomorphic case.

### 6.2.1 Restrictions to $T^*\mathbb{P}^1$ and $\mathbb{S}^2$

The above results extend naturally to the $T^*\mathbb{P}^1$ fiber sub-manifold. The point is that on this complex two dimensional non compact projective surface of $W\mathbb{P}^3(1,-1,1,-1)$,

one can repeat quite same steps. In doing so, one should restrict the previous covariant derivatives $\boldsymbol{\nabla}_{0,\pm}$ to the $T^*\mathbb{P}^1$ fiber; i.e

$$\boldsymbol{\nabla}_{0,\pm}|_{T^*\mathbb{P}^1} = \mathcal{D}_{0,\pm}, \tag{6.35}$$

and moreover impose conservation of the $\mathbb{C}^*$ charge which requires,

$$\mathcal{D}_0 = D_0 \qquad \Leftrightarrow \qquad A_0 = 0. \tag{6.36}$$

As such there is no gauge component $A_0$ and no $\sigma$ dependence. Furthermore, generic functions $G_n = G_n(x, y, z, w)$ living on $T^*\mathbb{P}^1$ obey the following eigenvalue eq,

$$2D_0 G_n = nG_n. \tag{6.37}$$

In particular, we have for the holomorphic gauge fields $A_\pm$ on $T^*\mathbb{P}^1$, the following eigenvalue eqs,

$$2D_0 A_\pm = \pm 2A_\pm. \tag{6.38}$$

Unlike the identity $\mathcal{D}_0 = D_0$, the two other gauge covariant derivatives $\mathcal{D}_\pm$ keep their original form,

$$\mathcal{D}_\pm = D_\pm - A_\pm D_0. \tag{6.39}$$

but with the gauge constraint eqs restricted to the fiber $T^*\mathbb{P}^1$,

$$\begin{aligned} [\mathcal{D}_+, \mathcal{D}_-] &= D_0, \\ [D_0, \mathcal{D}_+] &= \mathcal{D}_+ \\ [D_0, \mathcal{D}_-] &= -\mathcal{D}_- \end{aligned} \tag{6.40}$$

As such the previous three constraint relations (6.23) reduce to the following one,

$$(D_+A_- - D_-A_+) - 2A_+A_- = (\mathcal{D}_+A_- - \mathcal{D}_-A_+) = 0, \tag{6.41}$$

together with the obvious identities, $D_0A_\pm = \pm A_\pm$. By thinking about this constraint eq as a gauge field equation of motion following from minimizing a gauge invariant holomorphic field action $S_{T^*\mathbb{P}^1} = S_{T^*\mathbb{P}^1}[A_-, A_+, \Lambda_0]$ with respect to some some Lagrange field parameter $\Lambda_0$; that is

$$\frac{\delta S_{T^*\mathbb{P}^1}[A_-, A_+, \Lambda_0]}{\delta \Lambda_0} = 0, \tag{6.42}$$

one finds after integration,

$$S_{T^*\mathbb{P}^1} = \frac{1}{\lambda} \int_{T^*\mathbb{P}^1} d^2v \Lambda_0 (\mathcal{D}_+A_- - \mathcal{D}_-A_+), \tag{6.43}$$

where $d^2v$ is the holomorphic volume form on $T^*\mathbb{P}^1$. Clearly, this NC holomorphic $U(1)$ gauge field action $S_{T^*\mathbb{P}^1}$ is invariant under the gauge symmetry

$$A_\pm \to A_\pm + \mathcal{D}_\pm(\ln \Lambda). \tag{6.44}$$

One way to see it is by computing $\delta S_{T^*\mathbb{P}^1} \sim \int_{T^*\mathbb{P}^1} \Lambda_0 \left[\mathcal{D}_+, \mathcal{D}_-\right] (\ln \Lambda)$, which vanishes identically due to the identity $D_0 (\ln \Lambda) = 0$. The reduction down to $\mathbb{S}^2$ follows directly by imposing the reality condition. We get

$$S_{\mathbb{S}^2} = \frac{1}{\lambda_{CS}} \int_{\mathbb{S}^2} d^2 v \Lambda_0 \left(\mathcal{D}_+ C_- - \mathcal{D}_- C_+\right) \tag{6.45}$$

where we have no gauge component $C_0$ and where $d^2 v$ stands for the real volume 2-form of the two sphere.

### 6.2.2 Reductions to $T^*\mathbb{S}^1$ and $\mathbb{S}^1$

The above results may be also reduced down to the $T^*\mathbb{S}^1$ base sub-manifold. The gauge covariant derivatives on conifold $\boldsymbol{\nabla}_{0,\pm}$ when restricted to the base $T^*\mathbb{S}^1$ reduce to $\mathcal{T}_{0,\pm}$ with,

$$\mathcal{T}_\pm = T_\pm \qquad \Leftrightarrow \qquad A_\pm = 0, \tag{6.46}$$

and

$$\mathcal{T}_0 = T_0 - A_0 T_0, \tag{6.47}$$

where the gauge field $A_0$ has now no dependence of fiber variables, i.e,

$$A_0 = A_0 (\sigma) \,. \tag{6.48}$$

As such there is no gauge components $A_\pm$ and no $(x, y, z, w)$ dependence. The constraint eqs for gauge field on conifold reduce to,

$$T_\pm A_0 = 0, \tag{6.49}$$

and

$$A_0 = 0.$$

By equating this last relation with the action principle $\frac{\delta S_{T^*\mathbb{S}^1}}{\delta A_0} = 0$, we get the topological holomorphic action $S_{T^*\mathbb{S}^1} = S_{T^*\mathbb{S}^1} [A_0]$ on the base sub-manifold $T^*\mathbb{S}^1$ which reads then as,

$$\mathcal{S}_{T^*\mathbb{S}^1} = \frac{1}{\lambda} \int_{T^*\mathbb{S}^1} \frac{d\sigma}{\sigma} A_0^2, \tag{6.50}$$

where there is no kinetic term in agreement with the topological nature ot the theory. Clearly, this NC holomorphic $U(1)$ gauge field action $S_{T^*\mathbb{P}^1}$ is invariant under the gauge symmetry,

$$A_0 \rightarrow A_0 + \sigma \frac{\partial \left(\ln \Lambda\right)}{\partial \sigma}. \tag{6.51}$$

On the the unit circle, $\sigma = e^{i\theta}$ with $0 \leq \theta < 2\pi$, this topological action reduces to $\mathcal{S}_{\mathbb{S}^1} = \frac{1}{\lambda} \int_{\mathbb{S}^1} d\theta C_0^2$ which is just a constant.

# 7 Conclusion and outlook

In an attempt to look for new methods to approach NC topological gauge theories involving conifold background, we have first developed a way to study the link between conifold and non commutativity opening then a window for dealing with these backgrounds by borrowing q-deformed quantum mechanics methods. Then we have studied the explicit derivation of NC topological $SL(2)$ gauge theory on conifold and its sub-varieties using $T^*\mathbb{S}^3$ isometries. To do so, we have started by showing that conifold defining eq $xy - zw = \varepsilon^{ij}x_i y_j = \mu$ may be viewed as just the non trivial relation of the defining eqs

$$[z_I, z_J]_q = B_{IJ}, \qquad I, J = 1, 2, 3, 4, \tag{7.1}$$

of non commutative complex four dimension space; but with a very specific magnetic field eqs(2.8,3.3). In comparing our approach with known results in literature, we have noted striking similarities with Susskind way to approach quantum Hall systems and Ooguri-Vafa-Verlinde study of Hartle-Hawking Wave-Function for Flux Compactifications. We have developed the similarity with Susskind non commutative model for the Laughlin state of fractional quantum Hall system, known also to be described by a NC Chern Simons $U(1)$ gauge theory in $(2+1)$ dimensions. Then we have made a step in relating this feature to the attractor mechanism of [24] also known to have a strong link with non commutative geometry in so called mini-superspace.

Moreover, using the group factorisation of conifold $SL(2)$ isometry as $\mathbb{C}^* \times (SL(2)/\mathbb{C}^*)$, we have developed the corresponding projective hypersurface representation of conifold. This way of doing has the advantage of being directly related to the moduli space of supersymmetric theories whose simplest model, with a $U_{gauge}(1) \times SU_{global}(2)$ symmetry, has been constructed in section 4. The holomorphic superpotential $\mathcal{L}$ of this supersymmetric model involves the set of chiral matter $(Q_{+\alpha}, P_{-\beta}, \Sigma_+, \Sigma_-, \Phi, \Upsilon)$, see also (4.17). The holomorphic eqs of motion read as

$$\begin{aligned} Q_{+\alpha}P_{-\beta}\varepsilon^{\alpha\beta} + \Sigma_+\Sigma_- &= \mu, \\ \Sigma_+\Sigma_- &= 1, \end{aligned} \tag{7.2}$$

where, for simplicity, we have dropped out the coupling constants. The apparition of the charged superfields $\Sigma_+$ and $\Sigma_-$ is one of the predictions of this construction.

Furthermore, thinking about conifold as a projective complex three dimension hypersurface embedded in non compact $W\mathbb{P}^5(1, -1, 1, -1, 1, -1)$, we have developed a method to get topological gauge theory by focusing on the gauging the $\mathbb{C}^*$ projective isometry. We have also studied the reduction of $SL(2)$ gauge model on conifold down to its complex two and one dimensions sub-manifold $T^*\mathbb{P}^1$ and $T^*\mathbb{S}^1$; as well as their real

slices. Details on these topological gauge reductions are exposed in section 6.

In the end of this study, we would like to add a comment on higher dimension extensions of these geometries. As far as non commutative structure is concerned, conifold results may be extended to higher complex geometries by using the method presented in this paper. A direct extension concerns complex dimension $(4n-1)$ symplectic manifolds $SO(4n,\mathbb{C})/SO(4n-1,\mathbb{C})$ describing the hypersurface $\sum_{a=1}^{n}(x_a y_{a+n} - x_{a+n} y_a = \mu)$ embedded in $\mathbb{C}^{4n}$. This equation may be put in the form and reads also as,

$$\sum_{A,B=1}^{2n} \Omega^{AB}(x_A y_B - x_B y_A) = \mu, \tag{7.3}$$

where $\Omega^{AB}$ is the usual antisymmetric tensor of symplectic spinors. Following the method outlined in section 3, this relation may be also put as $x_A y_B - x_B y_A \sim \mu\Omega_{AB}$ or equivalently

$$x_A y_B - \mathcal{R}_{AB}^{CD} y_C x_D \sim \mu\Omega_{AB} \tag{7.4}$$

describing the link between these geometries and q-deformed non commutative geometry in complex $4n$ dimensions with a constant magnetic field $\mu\Omega_{AB}$. The particular case $n = 1$ corresponds to conifold geometry discussed in this paper. The next geometry, namely

$$(x_1 y_3 - x_3 y_1) + (x_2 y_4 - x_4 y_2) = \mu, \tag{7.5}$$

has a complex dimension 7, containing $\mathbb{S}^7$ as a real slice, and a manifest $SP(1,\mathbb{C})$ isometry subgroup rotating the symplectic spinor $x_a$ into $y_b$ and vice versa. This construction could be relevant for M-theory compactifications and $G_2$ manifolds.

An other interesting extension deals with complex dimension $(n^2-1)$ embedded in $\mathbb{C}^{n^2}$ parameterized by $n \times n$ matrix $Z$. These geometries have a $SL(n,C)$ isometry group and are described by the holomorphic order $n$ polynomial equation,

$$\det Z = \mu. \tag{7.6}$$

This algebraic relation captures a kind of generalized q-deformed non commutative structure à la Nambu bracket aiming the construction of generalizations of the hamiltonian mechanics based on Poisson bracket and usual commutator. Indeed, using the $n$ dimensional completely antisymmetric tensor $\varepsilon_{i_1 \ldots i_n}$, we can bring the above equation to the remarkable form,

$$Z_{1[j_1} Z_{2j_2} \ldots Z_{nj_n]} = -\frac{\mu}{N!}\varepsilon_{i_1 \ldots i_n}, \tag{7.7}$$

and all others vanish identically. To fix the ideas, one may consider the complex eight algebraic geometry with $SL(3,C)$ isometries. This geometry has some particularizes. First it may be directly related to non commutative extension of Nambu mechanics

whose bracket is associated with the determinant of real $3 \times 3$ matrices [38]. Second, it could be also relevant for type II superstring and M-theory compactificactions. Let us give some explicit details concerning this specific example. Consider the $3 \times 3$ complex holomorphic matrix coordinate,

$$Z_{ij} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \tag{7.8}$$

with $u_i = Z_{1i}$, $v_i = Z_{2i}$ and $w_i = Z_{3i}$. The natural extension of conifold geometry is given by the complex eight dimension hypersurface embedded in $\mathbb{C}^9$,

$$\det Z = \varepsilon^{ijk} u_i v_j w_k = \mu, \tag{7.9}$$

where $\varepsilon^{ijk}$ is the invariant three dimensional completely antisymmetric tensor. Like in case of conifold, this geometry is also singular for $\mu = 0$. In this coordinate system, the global isometry group is generated by,

$$\begin{aligned} D_1 &= u^i \frac{\partial}{\partial w^i}, \qquad D_{-1} = w^i \frac{\partial}{\partial u^i},, \qquad h_1 = [D_1, D_{-1}], \\ D_2 &= v^i \frac{\partial}{\partial w^i}, \qquad D_{-2} = w^i \frac{\partial}{\partial v^i}, \qquad h_2 = [D_2, D_{-2}], \\ D_3 &= u^i \frac{\partial}{\partial v^i}, \qquad D_{-3} = v^i \frac{\partial}{\partial v^i}, \qquad h_3 = [D_3, D_{-3}] = h_1 - h_2, \end{aligned} \tag{7.10}$$

and the link to non commutative geometry à la Nambu is given by,

$$\begin{aligned} u_{[i} v_j w_{k]} &= -\frac{\mu}{6} \varepsilon_{ijk}, \\ u_{[i} u_j w_{k]} &= u_{[i} v_j v_{k]} = u_{[i} w_j w_{k]} = 0. \end{aligned} \tag{7.11}$$

Concerning lower dimension sub-manifolds of eq(7.9), there are many; the natural one is obtained by reducing $SL(3, \mathbb{C})$ to $SL(3, \mathbb{R})$ or other sub-groups such as $SU(3, \mathbb{C})$ or also $T^*\mathbb{S}^1 \times T^*\mathbb{S}^3$. An other class of sub-manifolds is given by the following complex five dimension geometry,

$$\sum_{i=1}^{3} u_i \mu^i = \mu, \tag{7.12}$$

with

$$(v_j w_k - v_k w_j) = -\frac{1}{2} \mu^i \varepsilon_{ijk}, \tag{7.13}$$

where $\mu$ and $\mu^i$ are four constant numbers and where one recognizes conifolds block as sub-geometries.

Following the method we have developed in section 4, one may also build sub-manifolds with projective symmetries by help of the fibrations

$$SL(3) = \mathbb{C}^* \times (SL(3)/\mathbb{C}^*), \qquad SL(3) = \mathbb{C}^{*2} \times \left(SL(3)/\mathbb{C}^{*2}\right). \tag{7.14}$$

Using the group factorisation $SL(3) = \mathbb{C}^* \times (SL(3)/\mathbb{C}^*)$, one can introduce the projective $(\sigma; x_i, y_j, z_k)$ related to the old ones $(u_i, v_j, w_k)$ as follows,

$$x_i = \sigma^q u_i, \qquad y_j = \sigma^p v_i, \qquad z_k = \sigma^r w_i,, \qquad q + p + r = 0. \tag{7.15}$$

with the condition $q + p + r = 0$ and, to fix the ideas, can be chosen as $q = p = 1$, $r = -2$. This background has a natural supersymmetric quiver $\mathrm{QFT}_4$ realization extending directly the model we have given for conifold. For instance, the first relation of the superfield eqs of motion (7.2) extends directly as

$$\begin{aligned} X_{+i}Y_{+j}Z_{-2k}\varepsilon^{ijk} + \Sigma_+\Sigma_- &= \mu, \\ \Sigma_+\Sigma_- &= 1, \end{aligned} \tag{7.16}$$

with $\Sigma_+\Sigma_- = 1$ associated with $T^*\mathbb{S}^1$ and $X_{+i}Y_{+j}Z_{-2k}\varepsilon^{ijk} = \mu$ with the coset $SL(3)/\mathbb{C}^*$. Quite similar relations may written down for the others. For the fibration $SL(3) = \mathbb{C}^{*2} \times (SL(3)/\mathbb{C}^{*2})$, one may also build the projective hypersurface and the corresponding supersymmetric $\mathrm{QFT}_4$ realization by following the same method. The projective coordinates $(\sigma, \tau; x_i, y_j, z_k)$ are related to the old ones $(u_i, v_j, w_k)$ as follows,

$$x_i = \sigma^{q_1}\tau^{q_2}u_i, \qquad y_j = \sigma^{p_1}\tau^{p_2}v_i, \qquad z_k = \sigma^{r_1}\tau^{r_2}w_i, \tag{7.17}$$

with the two projective conditions $q_a + p_a + r_a = 0$. Under the scaling symmetries $\sigma \to \lambda_1\sigma$ and $\tau \to \lambda_2\tau$, the new coordinates transforms as,

$$x_i \to \lambda_1{}^{q_1}\lambda_2^{q_2}x_i, \qquad y_j \to \lambda_1{}^{p_1}\lambda_2{}^{p_2}y_i, \qquad z_k \to \lambda_1{}^{r_1}\lambda_2{}^{r_2}z_k \tag{7.18}$$

To build the corresponding $U_{gauge}(1) \times U_{gauge}(1) \times SU_{global}(3)$ supersymmetric quiver gauge theory, one should specify the solution of the constraint eqs $q_a + p_a + r_a = 0$ and follows the same line as in the $SL(2)$ case developed previously. In the special case $q_a + p_a = r_a = 0$, the superfield degrees of freedom extending (4.17) are summarized in the table below,

| 4D $\mathcal{N} = 1$ Superfields | $U(1) \times U(1) \times SU(3)$ |
|---|---|
| $V_a = -\theta\sigma^\mu\overline{\theta}A_\mu + \ldots, \quad a = 1, 2$ | $(0, 0, 1)$ |
| $\Phi_a = \phi_a + \theta\psi_a + \theta^2\mathrm{F}_a$ | $(0, 0, 1)$ |
| $X_i = x_i + \theta\chi_i + \theta^2F_i$ | $(1, -1, 3)$ |
| $Y_i = y_i + \theta\chi_i' + \theta^2F_i'$ | $(-1, 1, 3)$ |
| $Z_i = z_i + \theta\chi_i'' + \theta^2F_i''$ | $(0, 0, 3)$ |
| $\Sigma_{\pm1} = \sigma_{\pm1} + \theta\eta_{\pm1} + \theta^2L_{\pm1}$ | $(\pm1, 0, 1)$ |
| $\Sigma_{\pm2} = \sigma_{\pm2} + \theta\eta_{\pm2} + \theta^2L_{\pm2}$ | $(0, \pm1, 1)$ |
| $\Upsilon_{0a} = \gamma_{0a} + \theta\tau_{0a} + \theta^2G_{0a}$ | $(0, 0, 1)$ |

(7.19)

Other superfield configurations are also possible. We end this discussion by noting that it would be interesting to explore further the sub-manifolds of eqs(7.5,7.9) and look if they could be related with $G_2$ manifolds.

**Acknowledgement 2** *This work has been initiated during my visit to ICTP. I would like to thank the Senior Associate program of ICTP for kind hospitality and generosity. This research work is supported by the program Protars III D12/25, CNRST.*

# References

[1] Edward Witten, Ground Ring Of Two Dimensional String Theory, Nucl.Phys. B373 (1992) 187-213, hep-th/9108004,
Edward Witten, Barton Zwiebach, Algebraic Structures and Differential Geometry in 2D String Theory, Nucl.Phys. B377 (1992) 55-112, hep-th/9201056.

[2] Igor R. Klebanov, String Theory in Two Dimensions, lectures at the 1991 ICTP Spring School; hep-th/9108019

[3] Gregory Moore, M. Ronen Plesser, Sanjaye Ramgoolam, Exact S-Matrix for 2D String Theory, Nucl.Phys. B377 (1992) 143-190, hep-th/9111035

[4] R. Dijkgraaf, G. Moore, R. Plesser, The partition function of 2d string theory, Nucl.Phys. B394 (1993) 356-382, hep-th/9208031

[5] Debashis Ghoshal, Cumrun Vafa, c=1 String as the Topological Theory of the Conifold, Nucl.Phys. B453 (1995) 121-128, hep-th/9506122

[6] Mina Aganagic, Ken Intriligator, Cumrun Vafa, Nicholas P. Warner The Glueball Superpotential, Adv.Theor.Math.Phys. 7 (2004) 1045-1101, hep-th/0304271

[7] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa, D. Zanon, Perturbative Computation of Glueball Superpotentials, Phys.Lett. B573 (2003) 138-146, hep-th/0211017

[8] Freddy Cachazo, Cumrun Vafa, N=1 and N=2 Geometry from Fluxes, hep-th/0206017

[9] F. Cachazo, B. Fiol, K. Intriligator, S. Katz, C. Vafa, A Geometric Unification of Dualities, Nucl.Phys. B628 (2002) 3-78, hep-th/0110028

[10] R. Ahl Laamara, M. Ait Ben Haddou, A Belhaj, L.B Drissi, E.H Saidi, RG Cascades in Hyperbolic Quiver Gauge Theories, Nucl.Phys. B702 (2004) 163-188, hep-th/0405222

[11] F. Cachazo, S. Katz, C. Vafa, Geometric Transitions and N=1 Quiver Theories, hep-th/0108120

[12] Mina Aganagic, Andrew Neitzke, Cumrun Vafa, BPS Microstates and the Open Topological String Wave Function, hep-th/0504054

[13] Sergei Gukov, Albert Schwarz, Cumrun Vafa, Khovanov-Rozansky Homology and Topological Strings, hep-th/0412243

[14] Marcos Marino, Chern-Simons Theory and Topological Strings, Rev.Mod.Phys. 77 (2005) 675-720, hep-th/0406005

[15] Harald Ita, Harald Nieder, Yaron Oz, Tadakatsu Sakai, Topological B-Model, Matrix Models, $\hat{c}=1$ Strings and Quiver Gauge Theories, JHEP 0405 (2004) 058, hep-th/0403256

[16] Robbert Dijkgraaf, Cumrun Vafa, Matrix Models, Topological Strings, and Supersymmetric Gauge Theories, Nucl.Phys. B644 (2002) 3-20, hep-th/0206255

[17] Giulio Bonelli, Alessandro Tanzini, Maxim Zabzine, On topological M-theory, hep-th/0509175

[18] Nathan Seiberg, Edward Witten, String Theory and Noncommutative Geometry, JHEP 9909 (1999) 032, hep-th/9908142

[19] Mina Aganagic, Hirosi Ooguri, Natalia Saulina, Cumrun Vafa, Black Holes, q-Deformed 2d Yang-Mills, and Non-perturbative Topological Strings, Nucl.Phys. B715 (2005) 304-348, hep-th/0411280

[20] A. Belhaj, M. Hssaini, E. L. Sahraoui, E. H. Saidi, Explicit Derivation of Yang-Mills Self-Dual Solutions on non-Commutative Harmonic Space, Class.Quant.Grav. 18 (2001) 2339-2358, hep-th/0007137

[21] David Berenstein, Robert G. Leigh, Non-Commutative Calabi-Yau Manifolds, Phys.Lett. B499 (2001) 207-214,hep-th/0009209,
Adil Belhaj, El Hassan. Saidi, On Non Commutative Calabi-Yau Hypersurfaces , Phys.Lett. B523 (2001) 191-198, hep-th/0108143, NC Calabi-Yau Orbifolds in Toric Varieties with Discrete, .Phys. A38 (2005) 721-748, hep-th/0210167

[22] Mohamed Bennai, El Hassan Saidi, Toric Varieties with NC Toric Actions: NC Type IIA Geometry, Nucl.Phys. B677 (2004) 587-613, hep-th/0312200, NC Calabi-Yau Manifolds in Toric Varieties with NC Torus fibration, Phys.Lett. B550 (2002) 108-116, hep-th/0210073

[23] El Hassan Saidi, NC Geometry and Fractional Branes, Class.Quant.Grav. 20 (2003) 4447-4472, hep-th/0311245

[24] Hirosi Ooguri, Cumrun Vafa, Erik Verlinde, Hartle-Hawking Wave-Function for Flux Compactifications, hep-th/0502211

[25] Hirosi Ooguri, Andrew Strominger, Cumrun Vafa, Black Hole Attractors and the Topological String, Phys.Rev. D70 (2004) 106007? hep-th/0405146

[26] L. Susskind, The Quantum Hall Fluid and Non-Commutative Chern Simons Theory, hep-th/0101029,

[27] Andrew Neitzke, Cumrun Vafa, Topological strings and their physical applications, hep-th/0410178

[28] Sergei Gukov, Kirill Saraikin, Cumrun Vafa, A Stringy Wave Function for an $S^3$ Cosmology, hep-th/0505204

[29] Conifold in harmonic space and correlations functions in stringy $S^3$ cosmology, Lab/UFR-HEP 0512, GNPHE/05012, VACBT/05012

[30] Malika Ait Benhaddou, El Hassan Saidi, Explicit Analysis of Kahler Deformations in 4D N=1 Supersymmetric Quiver Theories, Physics Letters B575(2003)100-110, hep-th/0307103

[31] Edward Witten,Perturbative Gauge Theory As A String Theory In Twistor Space, Commun.Math.Phys. 252 (2004) 189-258, hep-th/0312171

[32] A.El Rhalami, E.M. Sahraoui, E.H.Saidi, NC Branes and Hierarchies in Quantum Hall Fluids, JHEP 0205 (2002) 004, hep-th/0108096,

[33] Aziz El Rhalami, El Hassan Saidi, NC Effective Gauge Model for Multilayer FQH, JHEP 0210:039,2002, hep-th/0208144 States

[34] R. Ahl Laamara, A. ElRhalami, E.H Saidi, Work in progress

[35] Alexios P. Polychronakos, Quantum Hall states as matrix Chern-Simons theory, JHEP 0104 (2001) 011, hep-th/0103013

[36] Ahmed Jellal, El Hassan Saidi, Hendrik B. Geyer, A Matrix Model for $\nu_{k_1 k_2} = \frac{k_1+k_2}{k_1 k_2}$ Fractional Quantum Hall States, hep-th/0204248

[37] James Gates Jr, Ahmed Jellal, EL Hassan Saidi, Michael Schreiber, Supersymmetric Embedding of the Quantum Hall Matrix Model, JHEP 0411 (2004), hep-th/0410070 075

[38] Azzouz Awane, Introduction à la géometrie k-symplectique, Afr.J.Math.Phys.1:125-135,2004.

[39] R. Ahl Laamara, L.B Drissi, E.H Saidi, D-string fluid in conifold, I. Topological gauge model, Lab/UFR-PHE 0516